

THREE DIMENSIONAL VORTEX APPROXIMATION CONSTRUCTION AND ε -LEVEL ESTIMATES FOR THE GINZBURG-LANDAU FUNCTIONAL

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ABSTRACT. We provide a quantitative three dimensional vortex approximation construction for the Ginzburg-Landau functional. This construction gives an approximation of vortex lines coupled to a lower bound for the energy, optimal to leading order, analogous to the 2D ones, and valid for the first time at the ε -level. These tools allow for a new approach to analyzing the behavior of global minimizers for the Ginzburg-Landau functional below and near the first critical field in 3D, followed in [Rom19]. In addition, they allow one to obtain an ε -quantitative product estimate for the study of Ginzburg-Landau dynamics.

Keywords: Ginzburg-Landau, free energy, first critical field, ε -level estimates, product estimate, vortices, vortex approximation construction, lower bound, vorticity estimate, minimal connections

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1. INTRODUCTION

1.1. The problem and a brief overview of the state of the art of the subject.

We are interested in studying the full three Ginzburg-Landau functional with applied magnetic field

$$GL_\varepsilon(u, A) = \frac{1}{2} \int_{\Omega} |\nabla_A u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 + \frac{1}{2} \int_{\mathbb{R}^3} |H - H_{\text{ex}}|^2,$$

which is a model for superconductors in a magnetic field.

Here

- Ω is a bounded domain of \mathbb{R}^3 , that we assume to be Lipschitz and simply connected.

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- $u : \Omega \rightarrow \mathbb{C}$ is called the *order parameter*. Its modulus squared (the density of Cooper pairs of superconducting electrons in the Bardeen-Cooper-Schrieffer (BCS) quantum theory) indicates the local state of the superconductor: where $|u|^2 \approx 1$ the material is in the superconducting phase, where $|u|^2 \approx 0$ in the normal phase.
- $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the electromagnetic vector potential of the induced magnetic field $H = \text{curl } A$.
- ∇_A denotes the covariant gradient $\nabla - iA$.
- $H_{\text{ex}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a given external (or applied) magnetic field.
- $\varepsilon > 0$ is the inverse of the *Ginzburg-Landau parameter* usually denoted κ , a non-dimensional parameter depending only on the material. We will be interested in the regime of small ε , corresponding to extreme type-II superconductors.

An essential feature of type-II superconductors is the occurrence of *vortices* (similar to those in fluid mechanics, but quantized) in the presence of an applied magnetic field. Physically, they correspond to normal phase regions around which a superconducting loop of current circulates. Since u is complex-valued, it can have zeros with a nonzero topological degree. Vortices are then *topological defects* of co-dimension 2 and are the crucial objects of interest in the analysis of the model.

Let us introduce the Ginzburg-Landau free energy

$$F_\varepsilon(u, A) = \frac{1}{2} \int_\Omega |\nabla_A u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 + |\text{curl } A|^2$$

and the Ginzburg-Landau energy without magnetic field

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2.$$

In the 1990's, mathematicians became interested in the Ginzburg-Landau model. In the pioneering work [BBH94] in the 2D setting (i.e. when Ω is assumed to be two dimensional), Bethuel, Brezis, and Hélein introduced systematic tools and asymptotic estimates to study vortices in the model without magnetic field, which is a complex-valued version of the Allen-Cahn model for phase transitions. A vortex in 2D is an object centered at an isolated zero of u , around which the phase of u has a nonzero winding number, called the degree of the vortex. A typical vortex centered at a point x_0 behaves like $u = \rho e^{i\varphi}$ with $\rho = f\left(\frac{|x-x_0|}{\varepsilon}\right)$, where $f(0) = 0$ and f tends to 1 as $r \rightarrow +\infty$, i.e. its characteristic core size is ε and

$$\frac{1}{2\pi} \int_{\partial B(x_0, R\varepsilon)} \frac{\partial \varphi}{\partial \tau} = d \in \mathbb{Z}$$

is its degree (also defined as the topological-degree of the map $u/|u| : \partial B(x_0, R\varepsilon) \rightarrow S^1$).

In [BBH94], the effect of the external magnetic field was replaced by a Dirichlet boundary condition $u = g$ on $\partial\Omega$, where g is an S^1 -valued map of degree $d > 0$. This boundary condition triggers the occurrence of vortices, allowing only for a fixed number of them. They proved that minimizers u of E_ε have d vortices of degree ± 1 and that

$$E_\varepsilon(u) \approx \pi d |\log \varepsilon| + W(a_1, \dots, a_d) \quad \text{as } \varepsilon \rightarrow 0,$$

where W is the “renormalized energy”, a function depending only on the vortex-centers a_i , which repel one another according to a Coulomb interaction. This analysis was then adapted to the study of the free energy by Bethuel and Rivière [BR95], under a Dirichlet boundary condition on $\partial\Omega$ that forces the presence of a fixed number of vortices.

A new approach was necessary to treat the case of the full model when the number of vortices gets unbounded as $\varepsilon \rightarrow 0$. Tools capable of handling this difficulty were developed after the works by Jerrard [Jer99] and Sandier [San98]. They introduced independently the ball construction method, which allows one to obtain universal lower bounds for two dimensional Ginzburg-Landau energies in terms of the topology of the vortices. These lower bounds capture the known fact that vortices of degree d cost at least an order $\pi |d| \log \frac{1}{\varepsilon}$ of energy. The second tool, which has been widely used in the analysis of the Ginzburg-Landau model in any dimension after the work by Jerrard and Soner [JS02], is the Jacobian or vorticity estimate. The vorticity is defined, for any sufficiently regular configuration (u, A) , as

$$\mu(u, A) = \text{curl}(iu, \nabla_A u) + \text{curl } A,$$

where (\cdot, \cdot) denotes the scalar product in \mathbb{C} identified with \mathbb{R}^2 i.e. $(a, b) = \frac{\bar{a}b + a\bar{b}}{2}$. This quantity is the $U(1)$ -gauge invariant version of the Jacobian determinant of u and is the analog of the vorticity of a fluid. The vorticity estimate allows one to relate the vorticity $\mu(u, A)$ with Dirac masses supported on co-dimension 2 objects, which in 2D are points naturally derived from the ball construction. In a series of works summarized in the book [SS07], Sandier and Serfaty analyzed the full two dimensional model and characterized the behavior of global minimizers of GL_ε in different regimes of the applied field (see also [SS00a, SS00b, SS00c, SS03]).

Rivière [Riv95], was the first to study the asymptotic behavior of minimizers of the free energy, under a Dirichlet boundary condition, as $\varepsilon \rightarrow 0$ in the 3D setting. Roughly speaking, vortices in 3D are small tubes of radius $O(\varepsilon)$ around the one dimensional zero-set of u . In the limit $\varepsilon \rightarrow 0$ vortices become curves L with an integer multiplicity d , whose cost is at least an order $\pi d |L| |\log \varepsilon|$ of energy, where $|L|$ denotes the length of L . In [Riv95], using an η -ellipticity result, Rivière identified the limiting one dimensional singular set of minimizers of F_ε with a mass minimizing current, which corresponds to a *minimal connection*. This concept was introduced in the work by Brezis, Coron, and Lieb [BCL86]. A new approach by Sandier in [San01],

combined the use of this object with a suitable slicing procedure to obtain the same result of Rivière in the case without magnetic field, and a generalization to higher dimension. We refer the interested reader to [LR99, BBO01, LR01, BBM04, ABO05, SS17] for further results in dimensions 3 and higher, when the applied magnetic field is zero.

Jerrard, Montero, and Sternberg [JMS04] established the existence of locally minimizing vortex solutions to the full three dimensional Ginzburg-Landau energy. Recently, Baldo, Jerrard, Orlandi, and Soner [BJOS12, BJOS13], via a Γ -convergence argument, described the asymptotic behavior of the full model as $\varepsilon \rightarrow 0$. We point out that conversely to the 2D situation, which is well understood, many questions remain open in 3D, in particular obtaining all the analogues of the 2D results contained in [SS07]. This is due to the more complicated geometry of the vortices in 3D, which have to be understood in the framework of currents and using geometric measure theory.

1.2. ε -level estimates for the Ginzburg-Landau functional. The key in Ginzburg-Landau analysis has proven to be a vortex approximation construction providing both approximation of the vorticity and lower bound. In 2D, this corresponds to the ball construction [San98, Jer99, SS07], which is a purely two dimensional method that provides ε -quantitative estimates. In 3D (and higher), based on the Federer-Fleming polyhedral deformation theorem, a not quantitative construction was provided in [ABO05] and later revisited in [BJOS12].

In this paper we present a new 3D vortex approximation construction, which provides an approximation of vortex filaments coupled to a lower bound for the energy, optimal to leading order, analogous to the 2D ones, and valid for the first time at the ε -level. Roughly speaking, our approximation is made as follows: for configurations $(u_\varepsilon, A_\varepsilon)$ whose free energy is bounded above by a suitable function of ε , we consider a grid of side-length $\delta = \delta(\varepsilon)$. If appropriately positioned, the grid can be taken to satisfy that $|u_\varepsilon| > 5/8$ on every edge of a cube. Then a 2D vorticity estimate implies that the restriction of the vorticity $\mu(u_\varepsilon, A_\varepsilon)$ to the boundary of every cube is well approximated by a linear combination of Dirac masses. Using minimal connections, we connect the points of support of these measures. Our choice of grid ensures a good compatibility between the objects constructed in cubes that share a face. Finally, by considering the distance

$$d_{\partial\Omega}(x, y) = \min\{|x - y|, d(x, \partial\Omega) + d(y, \partial\Omega)\},$$

we construct our approximation close to $\partial\Omega$, using minimal connections defined in terms of this distance. This process yields a closed polyhedral 1-dimensional current ν_ε , or, more precisely, a sum in the sense of currents of Lipschitz curves, that approximates well the vorticity $\mu(u_\varepsilon, A_\varepsilon)$ in a suitable norm.

We may now state our main results.

Theorem 1.1 (ε -level estimates for Ginzburg-Landau in 3D). *Assume that $\partial\Omega$ is of class C^2 . For any $m, n, M > 0$ there exist $C, \varepsilon_0 > 0$ depending only on m, n, M , and $\partial\Omega$, such that, for any $\varepsilon < \varepsilon_0$, if $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq M|\log \varepsilon|^m$ then there exists a polyhedral 1-dimensional current ν_ε such that*

- (1) ν_ε/π is integer multiplicity,
- (2) $\partial\nu_\varepsilon = 0$ relative to Ω ,
- (3) $\text{supp}(\nu_\varepsilon) \subset S_{\nu_\varepsilon} \subset \bar{\Omega}$ with $|S_{\nu_\varepsilon}| \leq C|\log \varepsilon|^{-q}$, where $q(m, n) := \frac{3}{2}(m + n)$,
- (4)

$$(1.1) \quad \int_{S_{\nu_\varepsilon}} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 + |\text{curl } A_\varepsilon|^2 \geq |\nu_\varepsilon|(\Omega) \left(\log \frac{1}{\varepsilon} - C \log \log \frac{1}{\varepsilon} \right) - \frac{C}{|\log \varepsilon|^n},$$

- (5) and for any $\gamma \in (0, 1]$ there exists a constant C_γ depending only on γ and $\partial\Omega$, such that

$$(1.2) \quad \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C_T^{0,\gamma}(\Omega)^*} \leq C_\gamma \frac{F_\varepsilon(u_\varepsilon, A_\varepsilon) + 1}{|\log \varepsilon|^{q\gamma}}.$$

Notation and definitions of the objects and spaces involved in this result can be found in the preliminaries (see Section 2).

Remark 1.1. *Alternatively, the right-hand sides of the lower bound and the vorticity estimate can be expressed in terms of the free energy $F_\varepsilon(u_\varepsilon, A_\varepsilon)$ of the configuration $(u_\varepsilon, A_\varepsilon)$ and a length $\delta = \delta(\varepsilon)$, which measures how “close” $\mu_\varepsilon(u_\varepsilon, A_\varepsilon)$ is to ν_ε , and which is a parameter of the construction (the side-length of the aforementioned grid). This will be done in the rest of the paper.*

We also remark that the right-hand side of (1.2) can be made small if $n > m \left(\frac{2}{3\gamma} - 1 \right)$.

The technical assumption that $\partial\Omega$ is of class C^2 allows us to find a lower bound for the free energy close to the boundary of the domain. If $\partial\Omega$ is only assumed to be Lipschitz, one has the following result.

Theorem 1.2. *For any $m, n, M > 0$ there exist $C, \varepsilon_0 > 0$ depending only on m, n , and M , such that, for any $\varepsilon < \varepsilon_0$, if $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq M|\log \varepsilon|^m$ then, letting $q := \frac{3}{2}(m + n)$ and defining*

$$\Omega_\varepsilon := \{x \in \Omega \mid d(x, \partial\Omega) \geq 2|\log \varepsilon|^{-q}\},$$

there exists a polyhedral 1-dimensional current ν_ε such that

- (1) ν_ε/π is integer multiplicity,
- (2) $\partial\nu_\varepsilon = 0$ relative to Ω ,

(3) $\text{supp}(\nu_\varepsilon) \subset S_{\nu_\varepsilon} \subset \bar{\Omega}$ with $|S_{\nu_\varepsilon}| \leq C |\log \varepsilon|^{-q}$,

(4)

$$\int_{S_{\nu_\varepsilon}} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 + |\text{curl } A_\varepsilon|^2 \geq |\nu_\varepsilon|(\Omega_\varepsilon) \left(\log \frac{1}{\varepsilon} - C \log \log \frac{1}{\varepsilon} \right) - \frac{C}{|\log \varepsilon|^n},$$

(5) and for any $\gamma \in (0, 1]$ there exists a constant C_γ depending only on γ and $\partial\Omega$, such that

$$\|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C_0^{0,\gamma}(\Omega)^*} \leq C_\gamma \frac{F_\varepsilon(u_\varepsilon, A_\varepsilon) + 1}{|\log \varepsilon|^{q\gamma}}.$$

As a direct consequence of Theorem 1.1, we recover and improve within our work setting, a well known result concerning the convergence as $\varepsilon \rightarrow 0$ of the vorticity of families of configurations whose free energy is bounded above by a constant times a power of $|\log \varepsilon|$. Results of the same kind can be found in [JS02, JMS04, SS04, ABO05, BJOS12].

Corollary 1.1. *Assume that $\partial\Omega$ is of class C^2 . Let $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ be a family of configurations of $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq M |\log \varepsilon|^m$ for some $m \geq 1$ and $M > 0$. Then, up to extraction,*

$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{|\log \varepsilon|^{m-1}} \rightharpoonup \mu \quad \text{in } C_T^{0,\gamma}(\Omega)^*$$

for any $\gamma \in (0, 1]$, where μ is a 1-dimensional current such that μ/π is integer multiplicity and $\partial\mu = 0$ relative to Ω . If $m = 1$ then μ is in addition rectifiable. Moreover,

$$\liminf_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(u_\varepsilon, A_\varepsilon)}{|\log \varepsilon|^m} \geq |\mu|(\Omega).$$

1.3. Application to the full Ginzburg-Landau functional. The behavior of global minimizers for GL_ε is determined by the strength of the external magnetic field H_{ex} . This model is known to exhibit several phase transitions, which occur for certain critical values of the intensity of H_{ex} . We are interested in the so-called *first critical field*, usually denoted by H_{c_1} . Physically, it is characterized as follows. Below H_{c_1} the superconductor is everywhere in its superconducting phase $|u| \approx 1$ and the external magnetic field is forced out by the material. This phenomenon is known as the *Meissner effect*. At H_{c_1} , which is of the order of $|\log \varepsilon|$ as $\varepsilon \rightarrow 0$, the first vortice(s) appear and the external magnetic field penetrates the material through the vortice(s).

In the works [Ser99, SS00a, SS03, SS07], Sandier and Serfaty derived with high precision the value of the first critical field and rigorously described the behavior of global minimizers of GL_ε below and near H_{c_1} in 2D. In the 3D setting, Alama, Bronsard, and Montero [ABM06] identified a candidate expression for H_{c_1} in the case of the ball. Then, Baldo, Jerrard, Orlandi, and Soner [BJOS13], characterized

to leading order the first critical field in 3D for a general bounded domain. In [Rom19], using crucially the ε -level estimates in Theorem 1.1, we computed H_{c_1} with higher precision and characterized the behavior of global minimizers for the full three dimensional Ginzburg-Landau energy below and near this value.

Since magnetic monopoles do not exist in Maxwell's electromagnetism theory, we may assume that $H_{\text{ex}} \in L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ is divergence-free. Then, there exists a vector potential $A_{\text{ex}} \in H^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$\text{curl } A_{\text{ex}} = H_{\text{ex}} \quad \text{and} \quad \text{div } A_{\text{ex}} = 0 \text{ in } \mathbb{R}^3.$$

Let us introduce the space

$$H_{\text{curl}} := \{A \in H^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3) \mid \text{curl } A \in L^2(\mathbb{R}^3, \mathbb{R}^3)\}.$$

The functional $GL_\varepsilon(u, A)$ is well defined for any pair $(u, A) \in H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$. The following result is a direct consequence of Theorem 1.1.

Corollary 1.2. *Theorem 1.1 holds true if the hypothesis that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F(u_\varepsilon, A_\varepsilon) \leq M|\log \varepsilon|^m$ is replaced with the assumption that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ is a configuration such that $GL_\varepsilon(u_\varepsilon, A_\varepsilon) \leq M|\log \varepsilon|^m$.*

Remark 1.2. *In particular, this result holds true if $(u_\varepsilon, A_\varepsilon)$ is a minimizing configuration for GL_ε in $H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ and $\|H_{\text{ex}}\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)}^2 \leq M|\log \varepsilon|^m$. Indeed, this follows by observing that*

$$GL_\varepsilon(u_\varepsilon, A_\varepsilon) = \inf_{H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]} GL_\varepsilon(u, A) \leq GL_\varepsilon(1, A_{\text{ex}}) = \int_{\Omega} |A_{\text{ex}}|^2 \leq C \|H_{\text{ex}}\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)}^2$$

for some universal constant C .

1.4. A quantitative product estimate for the study of Ginzburg-Landau dynamics. In this section, we consider the special case $\Omega = (0, T) \times \omega$, with $T > 0$ and $\omega \subset \mathbb{R}^2$, i.e. we deal with configurations $(u_\varepsilon, A_\varepsilon)$ which depend both on space and time. We use coordinates (t, x_1, x_2) in three-space and denote $\nabla = (\partial_{x_1}, \partial_{x_2})$, $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. We consider gauges of the form

$$A_\varepsilon = (\Phi_\varepsilon, B_\varepsilon),$$

where $\Phi_\varepsilon : (0, T) \times \omega \rightarrow \mathbb{R}$ and $B_\varepsilon : (0, T) \times \omega \rightarrow \mathbb{R}^2$. By using the notation

$$X^\perp = (-X_2, X_1), \quad \text{curl } X = \partial_{x_1} X_2 - \partial_{x_2} X_1$$

for vector fields X in the plane, we observe that

$$\text{curl } A_\varepsilon = (\text{curl } B_\varepsilon, \partial_t B_\varepsilon^\perp - \nabla^\perp \Phi_\varepsilon).$$

As above, the vorticity in three-space is defined by

$$\mu(u_\varepsilon, A_\varepsilon) := \text{curl}(\langle \partial_t u_\varepsilon - i\Phi_\varepsilon u_\varepsilon, iu_\varepsilon \rangle + \Phi_\varepsilon, \langle \nabla u_\varepsilon - iB_\varepsilon u_\varepsilon, iu_\varepsilon \rangle + B_\varepsilon).$$

This can be written as

$$\mu(u_\varepsilon, A_\varepsilon) = (J_\varepsilon, V_\varepsilon),$$

where

$$J_\varepsilon := \operatorname{curl}(\nabla u_\varepsilon, iu_\varepsilon) + \operatorname{curl}(1 - |u_\varepsilon|^2)B_\varepsilon$$

is the space-only vorticity and

$$V_\varepsilon := 2(i\partial_t u_\varepsilon, \nabla^\perp u_\varepsilon) + \partial_t(1 - |u_\varepsilon|^2)B_\varepsilon^\perp - \nabla^\perp(1 - |u_\varepsilon|^2)\Phi_\varepsilon$$

is the velocity. Since $\partial\mu(u_\varepsilon, A_\varepsilon)$ relative to Ω , we have the relation

$$\partial_t J_\varepsilon + \operatorname{div} V_\varepsilon = 0,$$

which means that the vorticity J_ε is transported by V_ε , hence the name velocity.

We let M_ε be a quantity such that

$$(1.3) \quad \forall q > 0, \quad \lim_{\varepsilon \rightarrow 0} M_\varepsilon \varepsilon^q = 0, \quad \lim_{\varepsilon \rightarrow 0} \frac{\log \varepsilon}{M_\varepsilon^q} = 0, \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \frac{\log M_\varepsilon}{\log \varepsilon} = 0.$$

For example $M_\varepsilon = e^{\sqrt{|\log \varepsilon|}}$ will do. Under the hypothesis of Theorem 1.2, our construction provides an approximation for the vorticity in three-space, which in particular yields an approximation for the velocity V_ε and for the space-only vorticity J_ε . This combined with ideas from [SS04, Ser17], yields a quantitative three dimensional product estimate, which allows one to control the velocity.

Theorem 1.3. *Assume ω to be Lipschitz and let M_ε be as above. Consider a function $f \in C_0^{0,1}([0, T] \times \bar{\omega})$ and a spatial vector field $X \in C_0^{0,1}([0, T] \times \bar{\omega})$. For any $m, n, M, \Lambda > 0$ with $n > \frac{1}{3}(2 - m)$ there exist a universal constant $C > 0$ and $\varepsilon_0 > 0$ depending only on m, n, M, Λ, f and X , such that, for any $\varepsilon < \varepsilon_0$, if $(u_\varepsilon, A_\varepsilon) = (u_\varepsilon, (\Phi_\varepsilon, B_\varepsilon)) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq M |\log \varepsilon|^m$ then*

$$\begin{aligned} & \int_{(0,T) \times \omega} \frac{|f|^2}{\Lambda} |\partial_t u_\varepsilon - iu_\varepsilon \Phi_\varepsilon|^2 + \int_{(0,T) \times \omega} \Lambda |X \cdot (\nabla u_\varepsilon - iu_\varepsilon B_\varepsilon)|^2 \\ & \geq (|\log \varepsilon| - C \log M_\varepsilon) \left| \int_{(0,T) \times \omega} f \nu_\varepsilon \wedge (-X_2 dx_1 + X_1 dx_2) \right| \\ & \quad - C \int_{(0,T) \times \omega} \max(|\nu_\varepsilon \wedge dx_1|, |\nu_\varepsilon \wedge dx_2|) + O\left(|\log \varepsilon|^{-\frac{1}{2}(m+3n)+1}\right), \end{aligned}$$

where ν_ε is the polyhedral 1-dimensional current ν_ε associated to $(u_\varepsilon, A_\varepsilon)$ by Theorem 1.2.

Remark 1.3. *By choosing*

$$\Lambda = \left(\frac{\int_{(0,T) \times \omega} |f|^2 |\partial_t u_\varepsilon - iu_\varepsilon \Phi_\varepsilon|^2}{\int_{(0,T) \times \omega} |X \cdot (\nabla u_\varepsilon - iu_\varepsilon B_\varepsilon)|^2} \right)^{1/2},$$

one obtains a left-hand side in the form of a product (plus error terms), hence the name product estimate. It is worth mentioning that the dependence of ε in terms of Λ , f , and X can be found in the proof (see Section 9).

1.5. A word about the proof of the main results. The subtle point of the proof is to obtain a lower bound for the free energy at the ε -level. Here is where minimal connections play a role. The idea of obtaining lower bounds for Ginzburg-Landau energies via the use of minimal connections was first introduced in [San01], in the case of the energy without magnetic field $E_\varepsilon(u)$. When trying to apply this kind of method to obtain lower bounds for the full functional $GL_\varepsilon(u, A)$, the main obstacle is that as soon as the external magnetic field is of the order of the first critical field, the number of vortices is a priori unbounded as $\varepsilon \rightarrow 0$. The main challenge in getting a lower bound that works at the ε -level is thus to keep track of the dependence of all the estimates on ε and $\delta(\varepsilon)$, taking into account that the number of vortex filaments may be unbounded.

Our method goes as follows: the choice of grid allows us to show that the restriction of the vorticity to the boundary of a cube \mathcal{C} can be well approximated by

$$2\pi \left(\sum_{i=1}^k \delta_{p_i} - \sum_{i=1}^k \delta_{n_i} \right),$$

where the points p_i 's are the (non-necessarily distinct) positive singularities and the points n_i 's are the (non-necessarily distinct) negative singularities. We remark that the number of points and their locations depend on ε .

By [BCL86], we know that there exists a 1-Lipschitz function $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that

$$\sum_{i=1}^k \zeta(p_i) - \sum_{i=1}^k \zeta(n_i) = L(\mathcal{A}),$$

where $L(\mathcal{A})$ is the length of the minimal connection associated to the configuration of points $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$. Since $|\nabla\zeta| \leq 1$, the co-area formula gives

$$\int_{\mathcal{C}} e_\varepsilon(u_\varepsilon) \geq \int_{\mathcal{C}} e_\varepsilon(u_\varepsilon) |\nabla\zeta| \geq \int_{t \in \mathbb{R}} \int_{\Sigma_t} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2 dt,$$

where $e_\varepsilon(u_\varepsilon) = \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2} (1 - |u_\varepsilon|^2)^2$ and $\Sigma_t = \{\zeta = t\} \cap \mathcal{C}$.

At this point, a vortex ball construction on a surface is necessary. Roughly speaking, if Σ_t is nice enough and $|u_\varepsilon| \geq 1/2$ on $\partial\Sigma_t$, we expect

$$\int_{\Sigma_t} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2 \geq \pi \deg(u_\varepsilon/|u_\varepsilon|, \partial\Sigma_t) \left(\log \frac{1}{\varepsilon} - O(\log |\log \varepsilon|) \right).$$

It turns out that, for most t 's, we have

$$\deg(u_\varepsilon/|u_\varepsilon|, \partial\Sigma_t) = \#\{i \mid \zeta(p_i) > t\} - \#\{i \mid \zeta(n_i) > t\}.$$

By noting that

$$\int_{t \in \mathbb{R}} \#\{i \mid \zeta(p_i) > t\} - \#\{i \mid \zeta(n_i) > t\} dt = \sum_{i=1}^k \zeta(p_i) - \sum_{i=1}^k \zeta(n_i) = L(\mathcal{A}) \approx \frac{1}{2\pi} |\nu_\varepsilon|(\mathcal{C}),$$

we are led to

$$\int_{\mathcal{C}} e_\varepsilon(u_\varepsilon) \geq \frac{1}{2} |\nu_\varepsilon|(\mathcal{C}) \left(\log \frac{1}{\varepsilon} - O(\log |\log \varepsilon|) \right) + \text{small error}.$$

Unfortunately, we cannot really use the function ζ in the previous argument, because its regularity is not sufficient to apply the ball construction on most of its level sets. To bypass this issue, we construct a smooth approximation of this function. The difficulties appear when trying to control the errors involved in the previously described method, because a quantitative bound of the second fundamental form of most of the level sets of our smooth approximation of the function ζ is needed.

In a similar but more involved way, by assuming that $\partial\Omega$ is of class C^2 , we can obtain a lower bound close to the boundary of the domain.

It is worth mentioning that by density arguments we can assume without loss of generality that u and A are of class C^1 in some proofs of this paper.

Outline of the paper. The paper is organized as follows.

In Section 2 we introduce some basic objects and spaces that are used throughout the paper, we recall some facts from the theory of currents and differential forms, and we describe the choice of grid.

In Section 3 we provide the ball construction method on a surface, which is one of the key tools used to obtain the lower bound for the free energy.

In Section 4 we show a 2D vorticity estimate. The main difference with classical results of the same kind is the space in which we prove the estimate.

In Section 5 we start by reviewing the concept of minimal connection. Then, we introduce the function ζ and the function ζ for $d_{\partial\Omega}$, and state three technical propositions concerning quantitative smooth approximations of these functions. Finally, we present our 3D vortex approximation construction.

Section 6 is devoted to the proof of a lower bound for the energy without magnetic field in the union of cubes of the grid, while in Section 7 we provide a similar estimate near the boundary of the domain. In these proofs we crucially use the results of Section 3 and Section 5.

In Section 8 we present the proofs of Theorem 1.1 and Theorem 1.2, which use the lower bounds obtained in Section 6 and Section 7, as well as the 2D vorticity estimate of Section 4.

In Section 9 we prove the quantitative product estimate for the study of Ginzburg-Landau dynamics.

In Appendix A we construct a quantitative smooth approximation of the function ζ . In Appendix B and Appendix C, we present two different methods to do the same for the function ζ for $d_{\partial\Omega}$. These are the most technical parts of the paper.

2. PRELIMINARIES

It is useful to introduce certain concepts and notation from the theory of currents and differential forms. We recall that in Euclidean spaces vector fields can be identified with 1-forms. Indeed, the vector field $F = (F_1, F_2, F_3)$ can be identified with the 1-form $F_1 dx_1 + F_2 dx_2 + F_3 dx_3$. We use the same notation for both the vector field and the 1-form.

It is also convenient to recall that a vector field F satisfying the boundary condition $F \times \nu = 0$ on $\partial\Omega$, where hereafter ν denotes the outer unit normal to $\partial\Omega$, is equivalent to a 1-form F such that $F_T = 0$ on $\partial\Omega$. Here F_T denotes the tangential component of F on $\partial\Omega$.

We define the superconducting current of a pair $(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ as the 1-form

$$j(u, A) = (iu, d_A u) = \sum_{k=1}^3 (iu, \partial_k u - iA_k u) dx_k.$$

This is related to the vorticity $\mu(u, A)$ of a configuration (u, A) through

$$\mu(u, A) = dj(u, A) + dA.$$

Thus $\mu(u, A)$ is an exact 2-form in Ω acting on couples of vector fields $(X, Y) \in \mathbb{R}^3 \times \mathbb{R}^3$ with the standard rule that $dx_i \wedge dx_j(X, Y) = X_i Y_j - X_j Y_i$. It can also be seen as a 1-dimensional current, which is defined through its action on 1-forms by the relation

$$\mu(u, A)(\phi) = \int_{\Omega} \mu(u, A) \wedge \phi.$$

We recall that the boundary of a 1-current T relative to a set Θ , is the 0-current ∂T defined by

$$\partial T(\phi) = T(d\phi)$$

for all smooth compactly supported 0-form ϕ defined in Θ . In particular, an integration by parts shows that the 1-dimensional current $\mu(u, A)$ has zero boundary relative to Ω . We denote by $|T|(\Theta)$ the mass of a 1-current T in Θ .

For $\alpha \in (0, 1]$ we let $C^{0,\alpha}(\Omega)$ denote the space of 1-forms ϕ such that $\|\phi\|_{C^{0,\alpha}(\Omega)} < \infty$. $C_0^{0,\alpha}(\Omega)$ denotes the space of 1-forms $\phi \in C^{0,\alpha}(\Omega)$ such that $\phi = 0$ on $\partial\Omega$, while $C_T^{0,\alpha}(\Omega)$ denotes the space of 1-forms $\phi \in C^{0,\alpha}(\Omega)$ such that $\phi_T = 0$ on $\partial\Omega$. The symbol $*$ is used to denote the dual spaces.

We next recall the definition of topological degree.

Definition 2.1. Let Σ be a complete oriented surface in \mathbb{R}^3 . If $\Theta \subset \Sigma$ is a smooth domain, and the map $u : \Sigma \rightarrow \mathbb{C}$ does not vanish on $\partial\Theta$, we can define the degree $\deg(u/|u|, \partial\Theta)$ of u restricted to $\partial\Theta$ to be the winding number of the map $u/|u| : \partial\Theta \rightarrow S^1$.

We observe that, because Σ is assumed to be oriented, $\partial\Theta$ carries a natural orientation. In the case that $\partial\Theta$ is not smooth, the topological degree can still be defined by approximation.

Hereafter \mathcal{H}^d denotes the d -dimensional Hausdorff measure, for $d \in \mathbb{N}$. When meaningful, we sometimes use the notation

$$F_\varepsilon(u, A, \Theta) := \int_{\Theta} e_\varepsilon(u, A) d\mathcal{H}^2, \quad E_\varepsilon(u, \Theta) := \int_{\Theta} e_\varepsilon(u) d\mathcal{H}^2,$$

with $e_\varepsilon(u, A) := \frac{1}{2} |\nabla_A u|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2 + |\operatorname{curl} A|^2$, $e_\varepsilon(u) := \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2$.

2.1. Choice of grid. Let us fix an orthonormal basis (e_1, e_2, e_3) of \mathbb{R}^3 and consider a grid $\mathfrak{G} = \mathfrak{G}(a, \delta)$ given by the collection of closed cubes $\mathcal{C}_i \subset \mathbb{R}^3$ of side-length $\delta = \delta(\varepsilon)$ (conditions on this parameter are given in the lemma below). In the grid we use a system of coordinates with origin in $a \in \Omega$ and orthonormal directions (e_1, e_2, e_3) . From now on we denote by \mathfrak{R}_1 (respectively \mathfrak{R}_2) the union of all edges (respectively faces) of the cubes of the grid. We have the following lemma.

Lemma 2.1 (Choice of grid). *For any $\gamma \in (-1, 1)$ there exist constants $c_0(\gamma), c_1(\gamma) > 0$, $\delta_0(\Omega) \in (0, 1)$ such that, for any $\varepsilon, \delta > 0$ satisfying*

$$\varepsilon^{\frac{1-\gamma}{2}} \leq c_0 \quad \text{and} \quad c_1 \varepsilon^{\frac{1-\gamma}{4}} \leq \delta \leq \delta_0,$$

if $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq \varepsilon^{-\gamma}$ then there exists $b_\varepsilon \in \Omega$ such that the grid $\mathfrak{G}(b_\varepsilon, \delta)$ satisfies

$$(2.1a) \quad |u_\varepsilon| > 5/8 \quad \text{on} \quad \mathfrak{R}_1(\mathfrak{G}(b_\varepsilon, \delta)) \cap \Omega,$$

$$(2.1b) \quad \int_{\mathfrak{R}_1(\mathfrak{G}(b_\varepsilon, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^1 \leq C \delta^{-2} F_\varepsilon(u_\varepsilon, A_\varepsilon),$$

$$(2.1c) \quad \int_{\mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^2 \leq C \delta^{-1} F_\varepsilon(u_\varepsilon, A_\varepsilon),$$

where C is a universal constant.

Proof. First, let us observe that, by the Cauchy-Schwarz inequality and the co-area formula, we have

$$\begin{aligned} 4F_\varepsilon(u_\varepsilon, A_\varepsilon) &\geq \int_\Omega |\nabla|u_\varepsilon||^2 + \frac{1}{\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \\ &\geq \int_\Omega \frac{|\nabla|u_\varepsilon|| (1 - |u_\varepsilon|^2)}{\varepsilon} \\ &= \int_{t=0}^\infty \left(\int_{\{|u|=t\}} \frac{(1-t^2)}{\varepsilon} d\mathcal{H}^2 \right) dt. \end{aligned}$$

Define $T := \{t \in [5/8, 3/4] \mid \text{Area}(\{|u_\varepsilon| = t\}) \leq \varepsilon^\alpha\}$ for $\alpha := \frac{1-\gamma}{2}$. From the previous estimate we deduce that

$$|T| \geq 1/8 - C\varepsilon^{1-\alpha} F_\varepsilon(u_\varepsilon, A_\varepsilon),$$

where hereafter $C > 0$ denotes a universal constant that may change from line to line. It is easy to check that there exists a constant $c_0(\gamma) > 0$ such that $|T| > 0$ for any $\varepsilon > 0$ satisfying $\varepsilon^{\frac{1-\gamma}{2}} \leq c_0$.

Fix a point $a \in \Omega$ and choose $\delta_0 = \delta_0(\Omega) \in (0, 1)$ such that $\{a + [0, \delta]^3\} \subset \Omega$ for any $0 < \delta < \delta_0$. Fix $t_0 \in T$ and define

$$B_I := \{b \in \{a + [0, \delta]^3\} \mid \{|u_\varepsilon| = t_0\} \cap \mathfrak{R}_1(\mathfrak{G}(b, \delta)) \neq \emptyset\}.$$

Let us show that $|B_I| \leq C\delta\varepsilon^\alpha$. We define, for any $b \in \{a + [0, \delta]^3\}$,

$$\Gamma_1^b = b + \mathbb{R} \times (\delta\mathbb{Z})^2, \quad \Gamma_2^b = b + (\delta\mathbb{Z}) \times \mathbb{R} \times (\delta\mathbb{Z}), \quad \text{and} \quad \Gamma_3^b = b + (\delta\mathbb{Z})^2 \times \mathbb{R}.$$

In addition, we set

$$\pi_1(x, y, z) = (y, z), \quad \pi_2(x, y, z) = (x, z), \quad \text{and} \quad \pi_3(x, y, z) = (x, y).$$

Let us observe that, for any $b_1 \in \{a_1 + [0, \delta]\}$, we have

$$\int_{(b_2, b_3) \in \{(a_2, a_3) + [0, \delta]^2\}} \chi_{\{(b_2, b_3) \mid \Gamma_1^b \cap \{|u_\varepsilon| = t_0\} \neq \emptyset\}} \leq \text{Area}(\pi_1(\{|u_\varepsilon| = t_0\})).$$

Therefore,

$$\int_{b_1 \in \{a_1 + [0, \delta]\}} \int_{(b_2, b_3) \in \{(a_2, a_3) + [0, \delta]^2\}} \chi_{\{(b_2, b_3) \mid \Gamma_1^b \cap \{|u_\varepsilon| = t_0\} \neq \emptyset\}} \leq \delta \text{Area}(\pi_1(\{|u_\varepsilon| = t_0\})).$$

Analogously, we have

$$\int_{b_2 \in \{a_2 + [0, \delta]\}} \int_{(b_1, b_3) \in \{(a_1, a_3) + [0, \delta]^2\}} \chi_{\{(b_1, b_3) \mid \Gamma_2^b \cap \{|u_\varepsilon| = t_0\} \neq \emptyset\}} \leq \delta \text{Area}(\pi_2(\{|u_\varepsilon| = t_0\})).$$

and

$$\int_{b_3 \in \{a_3 + [0, \delta]\}} \int_{(b_1, b_2) \in \{(a_1, a_2) + [0, \delta]^2\}} \chi_{\{(b_1, b_2) \mid \Gamma_3^b \cap \{|u_\varepsilon| = t_0\} \neq \emptyset\}} \leq \delta \text{Area}(\pi_3(\{|u_\varepsilon| = t_0\})).$$

We deduce that

$$|B_I| \leq \delta \sum_{i=1}^3 \text{Area}(\pi_i(\{|u_\varepsilon| = t_0\})) \leq C\delta \text{Area}(\{|u_\varepsilon| = t_0\}) \leq C\delta\varepsilon^\alpha.$$

Let us now observe that, for any $b \in \{a + [0, \delta]^3\} \setminus B_I$,

$$\text{either } |u_\varepsilon| > t_0 \quad \text{or} \quad |u_\varepsilon| < t_0 \quad \text{on } \mathfrak{R}_1(\mathfrak{G}(b, \delta)) \cap \Omega.$$

We let

$$B_{II} := \{b \in \{a + [0, \delta]^3\} \setminus B_I \mid \{|u_\varepsilon| < t_0\} \cap \mathfrak{R}_1(\mathfrak{G}(b, \delta)) \neq \emptyset\}$$

and observe that, for every $b \in B_{II}$, we have $(1 - |u_\varepsilon|^2) \geq (1 - t_0^2)$. This implies that

$$\frac{(1 - t_0^2)^2}{4\varepsilon^2} |B_{II}| \leq \int_{b \in B_{II}} \int_{\mathfrak{R}_1(\mathfrak{G}(b, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^1 d\mathcal{L}(b) \leq F_\varepsilon(u_\varepsilon, A_\varepsilon),$$

and thus $|B_{II}| \leq C\varepsilon^2 F_\varepsilon(u_\varepsilon, A_\varepsilon)$.

Now, we define $B_{\text{good}} := \{a + [0, \delta]^3\} \setminus (B_I \cup B_{II})$. Observe that

$$|B_{\text{good}}| \geq \delta^3 - C(\delta\varepsilon^\alpha + \varepsilon^{2-\gamma})$$

and that there exists a constant $c_1 > 0$ such that $|B_{\text{good}}| \geq \delta^3/2$ for any $\varepsilon, \delta > 0$ satisfying $c_1\varepsilon^{\frac{\alpha}{2}} \leq \delta$. Moreover, for any $b \in B_{\text{good}}$, we have

$$|u_\varepsilon| > t_0 \quad \text{on } \mathfrak{R}_1(\mathfrak{G}(b, \delta)) \cap \Omega.$$

Next, using a mean value argument we choose $b = b_\varepsilon \in B_{\text{good}}$ in such a way that

$$\int_{\mathfrak{R}_n(\mathfrak{G}(b_\varepsilon, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^n \leq C\delta^{n-3} F_\varepsilon(u_\varepsilon, A_\varepsilon) \quad \text{for } n = 1, 2.$$

First, by [ABO05, Lemma 8.4] there exists $b_\varepsilon \in B_{\text{good}}$ such that, for $n = 1, 2$,

$$\int_{\mathfrak{R}_n(\mathfrak{G}(b_\varepsilon, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^n \leq \frac{2}{|B_{\text{good}}|} \int_{B_{\text{good}}} \int_{\mathfrak{R}_n(\mathfrak{G}(b, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^n d\mathcal{L}(b).$$

Second, arguing as in the proof of [ABO05, Lemma 3.11], we have

$$\frac{1}{\delta^3} \int_{\{a + [0, \delta]^3\}} \delta^{3-n} \int_{\mathfrak{R}_n(\mathfrak{G}(b, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^n d\mathcal{L}(b) = CF_\varepsilon(u_\varepsilon, A_\varepsilon) \quad \text{for } n = 1, 2.$$

Then, we deduce that

$$\int_{\mathfrak{R}_n(\mathfrak{G}(b_\varepsilon, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^n \leq C \frac{\delta^3}{|B_{\text{good}}|} \delta^{n-3} F_\varepsilon(u_\varepsilon, A_\varepsilon) \quad \text{for } n = 1, 2.$$

Recalling that $|B_{\text{good}}| \leq \delta^3/2$, the lemma follows. \square

From now on we drop the cubes of the grid $\mathfrak{G}(b_\varepsilon, \delta)$, given by Lemma 2.1, whose intersection with $\mathbb{R}^3 \setminus \Omega$ is non-empty. We also define

$$(2.2) \quad \Theta := \Omega \setminus \cup_{\mathcal{C}_l \in \mathfrak{G}} \mathcal{C}_l \quad \text{and} \quad \partial\mathfrak{G} := \partial(\cup_{\mathcal{C}_l \in \mathfrak{G}} \mathcal{C}_l).$$

Observe that, in particular, $\partial\Theta = \partial\mathfrak{G} \cup \partial\Omega$.

We remark that $\mathfrak{G}(b_\varepsilon, \delta)$ carries a natural orientation. The boundary of every cube of the grid will be oriented accordingly to this orientation. Each time we refer to a face ω of a cube \mathcal{C} , it will be considered to be oriented with the same orientation of $\partial\mathcal{C}$. If we refer to a face $\omega \subset \partial\mathfrak{G}$, then the orientation used is the same of $\partial\mathfrak{G}$.

3. THE BALL CONSTRUCTION METHOD ON A SURFACE

In this section we use the method of Jerrard introduced in [Jer99] in order to construct balls containing all the zeros of u on a surface. This allows us to obtain a lower bound for the energy without magnetic field. The construction given here follows the one made by Sandier in [San01] that corresponds to an adaptation of the method of Jerrard. More precisely, our result extends [San01, Proposition 3.5], which deals with the case of a bounded number of vortices, to the situation in which the number of vortices is (a priori) unbounded as $\varepsilon \rightarrow 0$.

Proposition 3.1. *Let $\tilde{\Sigma}$ be a complete oriented surface in \mathbb{R}^3 whose second fundamental form is bounded by 1. Let Σ be a bounded open subset of $\tilde{\Sigma}$. For any $m, M > 0$ there exists $\varepsilon_0(m, M) > 0$ such that, for any $\varepsilon < \varepsilon_0$, if $u_\varepsilon \in H^1(\Sigma, \mathbb{C})$ satisfies*

$$(3.1) \quad E_\varepsilon(u_\varepsilon, \Sigma) \leq M |\log \varepsilon|^m$$

and

$$(3.2) \quad |u(x)| \geq \frac{1}{2} \quad \text{if } \mathfrak{d}(x, \partial\Sigma) < 1,$$

where $\mathfrak{d}(\cdot, \cdot)$ denotes the distance function in $\tilde{\Sigma}$, then, letting d be the winding number of $u_\varepsilon/|u_\varepsilon| : \partial\Sigma \rightarrow S^1$ and $M_\varepsilon = M |\log \varepsilon|^m$, we have

$$E_\varepsilon(u_\varepsilon, \Sigma) \geq \pi |d| \left(\log \frac{1}{\varepsilon} - \log M_\varepsilon \right).$$

To prove Proposition 3.1 we follow almost readily the proofs of [Jer99] and [San01].

3.1. Main steps. Let us define the essential null set $S_E(u_\varepsilon)$ of u_ε to be the union of those connected components U_i of $\{x \mid |u_\varepsilon(x)| < 1/2\}$ such that $\deg(u_\varepsilon/|u_\varepsilon|, \partial U_i) \neq 0$.

In the rest of this section each time we refer to a ball B of radius r we mean a geodesic ball of radius r in $\tilde{\Sigma}$.

First, we include $S_E(u_\varepsilon)$ in the union of well-chosen disjoint “small” balls B_i of radii $r_i > \varepsilon$ such that

$$E_\varepsilon(u_\varepsilon, B_i) \geq \frac{r_i}{C\varepsilon},$$

where the constant C does not depend on the second fundamental form of Σ when it is assumed to be bounded by 1. This is possible according to the following lemma.

Lemma 3.1. *Under the hypotheses of Proposition 3.1, there exist $C, r_0 > 0$ such that, for any $\varepsilon > 0$, there exist disjoint balls B_1, \dots, B_k of radii r_i such that*

- (1) $r_i \geq \varepsilon$ for all $i \in \{1, \dots, k\}$.
- (2) $S_E(u) \subset \cup_i B_i$ and $B_i \cap S_E(u) \neq \emptyset$ for all $i \in \{1, \dots, k\}$.
- (3) For all $i \in \{1, \dots, k\}$,

$$E_\varepsilon(u_\varepsilon, B_i \cap \Sigma) \geq \frac{\min\{r_i, r_0, 1\}}{C\varepsilon}.$$

Then the proof involves dilating the balls B_i into balls B'_i by combining them with annuli. A lower bound for $E_\varepsilon(u_\varepsilon, B'_i)$ is obtained by combining the lower bound for $E_\varepsilon(u_\varepsilon, B_i)$ and a lower bound for $E_\varepsilon(u_\varepsilon, B'_i \setminus \overline{B}_i)$.

Lemma 3.2. *Under the hypotheses of Proposition 3.1, there exist $C, \varepsilon_0, r_0 > 0$ such that, for any $0 < \varepsilon < s < r < r_0$, if $B_r, B_s \subset \Sigma$ are two concentric balls and if $S_E(u_\varepsilon) \cap (B_r \setminus \overline{B}_s) = \emptyset$ then, letting $d := \deg(u_\varepsilon/|u_\varepsilon|, \partial B_r)$,*

$$E_\varepsilon(u_\varepsilon, B_r \setminus \overline{B}_s) \geq |d| \left(\Lambda_\varepsilon \left(\frac{r}{|d|} \right) - \Lambda_\varepsilon \left(\frac{s}{|d|} \right) \right),$$

where $\Lambda_\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a function that satisfies the following properties:

- (1) $\Lambda_\varepsilon(t)/t$ is decreasing.
- (2) $\sup_{t \in \mathbb{R}_+} \Lambda_\varepsilon(t)/t \leq 1/(C\varepsilon)$.
- (3) If $0 < \varepsilon < \varepsilon_0$ and $\varepsilon < t < r_0$ then

$$\left| \Lambda_\varepsilon(t) - \pi \log \frac{t}{\varepsilon} \right| \leq C.$$

By taking into consideration the following adaptation of [San01, Lemma 3.12], the proofs of the previous two lemmas are straightforward modifications of the proofs of [San01, Lemma 3.8] and of [San01, Lemma 3.9].

Lemma 3.3. *Let $S_t(x)$ denote the geodesic circle in $\tilde{\Sigma}$ of radius t centered at $x \in \Sigma$. Under the hypotheses of Proposition 3.1, there exist $C, \varepsilon_0, r_0 > 0$ such that, for any $x \in \Sigma$ and for any $\varepsilon, t > 0$ satisfying $\varepsilon < \varepsilon_0$ and $\varepsilon < t < r_0$, if $|u_\varepsilon| \leq 1$ on $S_t(x)$ then*

$$E_\varepsilon(u_\varepsilon, S_t(x)) \geq \pi m^2 \left(\frac{|d|}{t} - C \right)^+ + \frac{(1-m)^C}{C\varepsilon},$$

where $m := \inf_{y \in S_t} |u_\varepsilon(y)|$ and

$$d := \begin{cases} \deg(u_\varepsilon/|u_\varepsilon|, S_t(x)) & m \neq 0, \\ 0 & m = 0. \end{cases}$$

Proof. By observing that the constants r_0, r , and C involved in (B.8), (B.9), and (B.12) in the proof of [San01, Lemma 3.12] can be chosen independently of the second fundamental form of $\tilde{\Sigma}$ when it is assumed to be bounded by 1, then the proof is verbatim the same as that of [San01, Lemma 3.2]. \square

Lemma 3.1 and Lemma 3.2 allow one to prove the following result, whose proof is a straightforward modification of the proof of [San01, Proposition 3.10].

Proposition 3.2. *For any $\varepsilon > 0$, let $\{B_i\}_i$ be the family of balls of radii r_i given by Lemma 3.1. Let*

$$d_i := \begin{cases} \deg(u_\varepsilon/|u_\varepsilon|, \partial B_i) & \text{if } \bar{B}_i \subset \Sigma_\varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$t_0 := \min_{\{i \mid d_i \neq 0\}} \frac{r_i}{|d_i|} \quad (\text{with } t_0 := +\infty \text{ if } d_i = 0 \text{ for every } i).$$

Then, for any $t \geq t_0$, there exists a family of disjoint geodesic balls $B_1(t), \dots, B_{k(t)}(t)$ of radii $r_i(t)$ in $\tilde{\Sigma}$ such that

- (1) $S_E(u) \subset \cup_i B_i(t)$ and $S_E(u) \cap B_i(t) \neq \emptyset$ for all $i \in \{1, \dots, k(t)\}$.
- (2) For all $i \in \{1, \dots, k(t)\}$, if $\bar{B}_i(t) \subset \Sigma$ then $r_i(t) \geq t|d_i(t)|$, where

$$d_i(t) := \deg(u_\varepsilon/|u_\varepsilon|, \partial B_i(t)).$$

- (3) For all $i \in \{1, \dots, k(t)\}$,

$$E_\varepsilon(u_\varepsilon, B_i(t) \cap \Sigma) \geq \min\{r_i(t), r_0, 1\} \frac{\Lambda_\varepsilon(t)}{t}.$$

Proof of Proposition 3.1. We assume that $d \neq 0$, otherwise the result is trivial. Apply Lemma 3.1, call the resulting balls B_1, \dots, B_k , and call r_1, \dots, r_k their radii. From Lemma 3.1 and (3.1), we have

$$\min\{r_i, r_0, 1\} \leq C\varepsilon E_\varepsilon(u_\varepsilon, B_i \cap \Sigma) \leq C\varepsilon M_\varepsilon \leq C\varepsilon |\log \varepsilon|^m,$$

where throughout the proof $C = C(M) > 0$ denotes a constant that may change from line to line. We deduce that there exists $\varepsilon_0(m, M) > 0$ such that, for any $i \in \{1, \dots, k\}$ and for any $\varepsilon < \varepsilon_0$,

$$(3.3) \quad r_i = \min\{r_i, r_0, 1\} \leq C\varepsilon M_\varepsilon \quad \text{and} \quad r_i \leq \frac{1}{2}.$$

Since $\mathfrak{d}(S_E(u_\varepsilon), \partial\Sigma) < 1$, which follows from (3.2), and $B_i \cap S_E(u_\varepsilon) \neq \emptyset$, we conclude that $\overline{B}_i \subset \Sigma$. Thus

$$(3.4) \quad \sum_{i=1}^k \deg(u_\varepsilon/|u_\varepsilon|, B_i) = d \neq 0.$$

As in Proposition 3.2, let

$$t_0 = \min_{\{i \mid d_i \neq 0\}} \frac{r_i}{|d_i|}.$$

From (3.3) and (3.4), we get that $t_0 \leq C\varepsilon M_\varepsilon$. Fix $\alpha \in (0, 1)$ with $\alpha < m$. By reducing the constant ε_0 , we deduce that $t_0 \leq M_\varepsilon^{-1} |\log \varepsilon|^\alpha$ for any $\varepsilon < \varepsilon_0$. Therefore, we may apply Proposition 3.2 with $t = M_\varepsilon^{-1} |\log \varepsilon|^\alpha$. This yields balls $B_1(t), \dots, B_{k(t)}(t)$ with radii $r_i(t)$ and degrees $d_i(t)$ such that

$$\min\{r_i(t), r_0, 1\} \leq E_\varepsilon(u_\varepsilon, B_i(t) \cap \Sigma) \frac{t}{\Lambda_\varepsilon(t)}.$$

From Lemma 3.2, we have

$$r_i(t) = \min\{r_i(t), r_0, 1\} \leq M_\varepsilon \frac{M_\varepsilon^{-1} |\log \varepsilon|^\alpha}{C |\log \varepsilon|} \leq C |\log \varepsilon|^{\alpha-1}.$$

Once again, since $\mathfrak{d}(S_E(u_\varepsilon), \partial\Sigma) < 1$ (recall (3.2)) and $B_i(t) \cap S_E(u) \neq \emptyset$, by possibly further reducing the constant ε_0 , we deduce that $\overline{B}_i(t) \subset \Sigma$ for any $i \in \{1, \dots, k(t)\}$ and for any $\varepsilon < \varepsilon_0$. Hence $d = \sum_{i=1}^{k(t)} d_i(t)$. Then, from Proposition 3.2, $r_i(t) \geq t|d_i(t)|$ and therefore

$$E_\varepsilon(u_\varepsilon, \Sigma) \geq \sum_{i=1}^{k(t)} |d_i(t)| \Lambda_\varepsilon(t).$$

Since $\sum_{i=1}^{k(t)} |d_i(t)| \geq |d|$ and $t < r_0$ if ε is small enough (recall that $\alpha < m$), Lemma 3.2 implies that, for any $\varepsilon < \varepsilon_0$, for a possibly smaller ε_0 ,

$$E_\varepsilon(u, \Sigma) \geq \pi|d| \left(\log \frac{t}{\varepsilon} - C \right) \geq \pi|d| \left(\log \frac{1}{\varepsilon} - \log M_\varepsilon \right).$$

The proposition is proved. \square

Corollary 3.1. *Let $\tilde{\Sigma}$ be a complete oriented surface in \mathbb{R}^3 whose second fundamental form is bounded by $Q_\varepsilon = Q |\log \varepsilon|^q$, where $q, Q > 0$ are given numbers. Let Σ be a bounded open subset of $\tilde{\Sigma}$. For any $m, M > 0$ there exists $\varepsilon_0(m, q, M, Q) > 0$ such that, for any $\varepsilon < \varepsilon_0$, if $u_\varepsilon \in H^1(\Sigma, \mathbb{C})$ satisfies*

$$E_\varepsilon(u_\varepsilon, \Sigma) \leq M |\log \varepsilon|^m$$

and

$$(3.5) \quad |u(x)| \geq \frac{1}{2} \quad \text{if } \mathfrak{d}(x, \partial\Sigma) < Q_\varepsilon^{-1},$$

where $\mathfrak{d}(\cdot, \cdot)$ denotes the distance function in $\tilde{\Sigma}$, then, letting d be the winding number of $u_\varepsilon/|u_\varepsilon| : \partial\Sigma \rightarrow S^1$ and $\mathcal{M}_\varepsilon = M|\log \varepsilon|^m$ we have

$$E_\varepsilon(u_\varepsilon, \Sigma) \geq \pi|d| \left(\log \frac{1}{\varepsilon} - \log \mathcal{M}_\varepsilon Q_\varepsilon \right).$$

Proof. Let us consider the transformation

$$\tilde{u}_\varepsilon(y) = u_\varepsilon \left(\frac{y}{Q_\varepsilon} \right) \quad \text{for } y \in \Sigma_\varepsilon := Q_\varepsilon \Sigma.$$

We let $\tilde{\Sigma}_\varepsilon := Q_\varepsilon \tilde{\Sigma}$. Observe that, by a change of variables, we have

$$E_\varepsilon(u_\varepsilon, \Sigma) = E_{\tilde{\varepsilon}}(\tilde{u}_\varepsilon, \Sigma_\varepsilon),$$

where $\tilde{\varepsilon} := \varepsilon Q_\varepsilon$. It is easy to check that the second fundamental form of $\tilde{\Sigma}_\varepsilon$ is bounded by 1. Then a direct application of Proposition 3.1 shows that

$$E_\varepsilon(u_\varepsilon, \Sigma) = E_{\tilde{\varepsilon}}(\tilde{u}_\varepsilon, \Sigma_\varepsilon) \geq \pi|d| \left(\log \frac{1}{\tilde{\varepsilon}} - \log \mathcal{M}_\varepsilon \right) = \pi|d| \left(\log \frac{1}{\varepsilon} - \log \mathcal{M}_\varepsilon Q_\varepsilon \right)$$

for any $0 < \varepsilon < \varepsilon_1 = \varepsilon_0 Q_\varepsilon^{-1}$, where ε_0 is the constant appearing in the proposition. \square

4. A 2D VORTICITY ESTIMATE

Let ω be a two dimensional domain. For a given function $u : \omega \rightarrow \mathbb{C}$ and a given vector field $A : \omega \rightarrow \mathbb{R}^2$ we define

$$j(u, A) = (iu, \nabla_A u), \quad \mu(u, A) = dj(u, A) + dA.$$

We also let

$$F_\varepsilon(u, A, \omega) = \int_\omega e_\varepsilon(u, A), \quad F_\varepsilon(u, A, \partial\omega) = \int_{\partial\omega} e_\varepsilon(u, A) d\mathcal{H}^1,$$

where

$$e_\varepsilon(u, A) = |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 + |\operatorname{curl} A|^2.$$

We have the following 2D vorticity estimate.

Theorem 4.1. *Let $\omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. Let $u : \omega \rightarrow \mathbb{C}$ and $A : \omega \rightarrow \mathbb{R}^2$ be $C^1(\bar{\omega})$ and such that $|u| \geq 5/8$ on $\partial\omega$. Let $\{U_j\}_{j \in J}$ be the collection of connected components of $\{|1 - |u(x)|| \geq 1/2\}$ and $\{S_i\}_{i \in I}$ denote the collection of connected components of $\{|u(x)| \leq 1/2\}$ whose degree $d_i = \deg(u/|u|, \partial S_i) \neq 0$. Then, letting $r = \sum_{j \in J} \operatorname{diam}(U_j)$ and assuming $\varepsilon, r \leq 1$, we have*

$$(4.1) \quad \left\| \mu(u, A) - 2\pi \sum_{i \in I} d_i \delta_{a_i} \right\|_{C^{0,1}(\omega)^*} \leq C \max(\varepsilon, r) (1 + F_\varepsilon(u, A, \omega) + F_\varepsilon(u, A, \partial\omega)),$$

where a_i is the centroid of S_i and C is a universal constant.

Proof. As in [SS07, Chapter 6], we set $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ to be defined by

$$\begin{cases} \chi(x) = 2x & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \chi(x) = 1 & \text{if } x \in \left[\frac{1}{2}, \frac{3}{2}\right] \\ \chi(x) = 1 + 2\left(x - \frac{3}{2}\right) & \text{if } x \in \left[\frac{3}{2}, 2\right] \\ \chi(x) = x & \text{if } x \in [2, +\infty). \end{cases}$$

We then set $\tilde{u} : \omega \rightarrow \mathbb{C}$ by

$$\tilde{u}(x) = \frac{\chi(|u|)}{|u|}u$$

and let

$$\tilde{j} := (i\tilde{u}, d_A\tilde{u}), \quad \tilde{\mu} := d\tilde{j} + dA.$$

Step 1. Let us prove that

$$\|\mu(u, A) - \tilde{\mu}\|_{C^{0,1}(\omega)^*} \leq C\varepsilon(F_\varepsilon(u, A, \omega) + F_\varepsilon(u, A, \partial\omega)).$$

In fact, by integration by parts, for any function $\zeta \in C^{0,1}(\omega)$, we have

$$\left| \int_\omega \zeta(\mu(u, A) - \tilde{\mu}) \right| \leq \left| \int_\omega (\nabla\zeta)^\perp \cdot (j(u, A) - \tilde{j}) \right| + \left| \int_{\partial\omega} \zeta(j(u, A) - \tilde{j}) \cdot \vartheta^\perp \right|,$$

where ϑ is the outer unit normal to $\partial\omega$ and $x^\perp = (-x_2, x_1)$ for any vector $x = (x_1, x_2)$. Arguing as in [SS07, Lemma 6.2], we get

$$\begin{aligned} \left| \int_\omega (\nabla\zeta)^\perp \cdot (j(u, A) - \tilde{j}) \right| &\leq \|\nabla\zeta\|_{L^\infty(\omega)} \int_\omega \frac{||u|^2 - |\tilde{u}|^2|}{|u|} |\nabla_A u| \\ &\leq 3\|\nabla\zeta\|_{L^\infty(\omega)} \int_\omega |1 - |u|| |\nabla_A u| \\ &\leq C\|\nabla\zeta\|_{L^\infty(\omega)} \varepsilon F_\varepsilon(u, A, \omega). \end{aligned}$$

Since $|u| \geq 5/8$ on $\partial\omega$, a simple computation shows that

$$|j(u, A) - \tilde{j}| \leq 2(1 - |u|^2) |\nabla_A u| \quad \text{on } \partial\omega.$$

By the Cauchy-Schwarz inequality, we find

$$\begin{aligned} \left| \int_{\partial\omega} \zeta(j(u, A) - \tilde{j}) \cdot \vartheta^\perp \right| &\leq 2\|\zeta\|_{C^{0,1}(\omega)} \int_{\partial\omega} (1 - |u|^2) |\nabla_A u| d\mathcal{H}^1 \\ &\leq C\|\zeta\|_{C^{0,1}(\omega)} \varepsilon F_\varepsilon(u, A, \partial\omega). \end{aligned}$$

Thus,

$$\|\mu(u, A) - \tilde{\mu}\|_{C^{0,1}(\omega)^*} \leq C\varepsilon(F_\varepsilon(u, A, \omega) + F_\varepsilon(u, A, \partial\omega))$$

for some universal constant C .

Step 2. By Step 1, the proof reduces to showing that

$$\left\| \tilde{\mu} - 2\pi \sum_{i \in I} d_i \delta_{a_i} \right\|_{C^{0,1}(\omega)^*} \leq C \max(r, \varepsilon) (1 + F_\varepsilon(u, A, \omega) + F_\varepsilon(u, A, \partial\omega)).$$

Observe that $\tilde{j}+A = (i\tilde{u}, d\tilde{u})$ and $\tilde{\mu} = 0$ whenever $|\tilde{u}| = 1$, i.e. when $|1-|u|| \leq 1/2$. In particular

$$\int_{\{|1-|u|| \leq 1/2\}} \tilde{\mu} = 0.$$

On the other hand, for any $j \in J$, Stokes' theorem yields

$$\int_{U_j} \tilde{\mu} = \int_{\partial U_j} (i\tilde{u}, \nabla \tilde{u}) \cdot \tau = 2\pi d_j,$$

where $d_j = \deg(u/|u|, \partial U_j)$ and τ denotes the unit tangent vector to ∂U_j . By definition of the collection $\{S_i\}_{i \in I}$, we have

$$\sum_{j \in J} \int_{U_j} \tilde{\mu} = \sum_{i \in I} \int_{S_i} \tilde{\mu} = \sum_{i \in I} 2\pi d_i.$$

Let $\zeta \in C^{0,1}(\omega)$. From the previous computations, we deduce that

$$\int_{\omega} \zeta \tilde{\mu} = \sum_{j \in J} \int_{U_j} \zeta \tilde{\mu} = \sum_{i \in I} \zeta(a_i) \int_{S_i} \tilde{\mu} + \sum_{j \in J} \int_{U_j} (\zeta - \zeta(a_j)) \tilde{\mu},$$

where a_j denotes the centroid of U_j . Besides,

$$\sum_{i \in I} \zeta(a_i) \int_{S_i} \tilde{\mu} = 2\pi \sum_{i \in I} d_i \zeta(a_i) = 2\pi \sum_{i \in I} d_i \int_{\omega} \zeta \delta_{a_i}.$$

Let us note that, since ζ is a Lipschitz function, we have

$$|\zeta(x) - \zeta(a_j)| \leq \|\zeta\|_{C^{0,1}(\omega)} |x - a_j| \leq \|\zeta\|_{C^{0,1}(\omega)} \text{diam}(U_j)$$

for all $x \in U_j$.

On the other hand, observing that

$$\tilde{\mu} = 2(\partial_{x_1} \tilde{u} - iA_{x_1} \tilde{u}) \times (\partial_{x_2} \tilde{u} - iA_{x_2} \tilde{u}) + \text{curl } A,$$

we deduce that $|\tilde{\mu}| \leq 2|\nabla_A u|^2 + |\text{curl } A|$. Then, letting $F_\varepsilon(u, A, U_j) = \int_{U_j} e_\varepsilon(u, A)$,

the Cauchy-Schwarz inequality gives

$$\int_{U_j} |\tilde{\mu}| \leq 4 \left(F_\varepsilon(u, A, U_j) + |U_j|^{\frac{1}{2}} F_\varepsilon(u, A, U_j)^{\frac{1}{2}} \right).$$

Observe that $|U_j| \leq C \text{diam}(U_j)^2$. Collecting our previous computations, we find

$$\left| \sum_{i \in I} \int_{S_i} (\zeta - \zeta(a_i)) \tilde{\mu} \right| \leq Cr \|\zeta\|_{C^{0,1}(\omega)} \left(F_\varepsilon(u, A, \omega) + r F_\varepsilon(u, A, \omega)^{\frac{1}{2}} \right).$$

Remembering that $\sqrt{x} \leq 1 + x$, we get

$$\left| \int_{\omega} \zeta \tilde{\mu} - 2\pi \sum_{i \in I} d_i \int_{\omega} \zeta \delta_{a_i} \right| \leq Cr \|\zeta\|_{C^{0,1}(\omega)} (1 + F_\varepsilon(u, A, \omega)).$$

This concludes the proof of (4.1). \square

Given a two dimensional Lipschitz domain $\omega \subset \Omega$, we let $(s, t, 0)$ denote coordinates in \mathbb{R}^3 such that $\omega \subset \{(s, t, 0) \in \Omega\}$. We define $\mu_\varepsilon := \mu_\varepsilon(u, A)[\partial_s, \partial_t]$, and write $\mu_{\varepsilon, \omega}$ its restriction to ω . Theorem 4.1 immediately yields the following corollary.

Corollary 4.1. *Let $\gamma \in (0, 1)$ and assume that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq \varepsilon^{-\gamma}$, so that by Lemma 2.1 there exists a grid $\mathfrak{G}(b_\varepsilon, \delta)$ satisfying (2.1). Then there exists $\varepsilon_0(\gamma)$ such that, for any $\varepsilon < \varepsilon_0$ and for any face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, \delta))$ of a cube of the grid $\mathfrak{G}(b_\varepsilon, \delta)$, letting $\{U_{j, \omega}\}_{j \in J_\omega}$ be the collection of connected components of $\{x \in \omega \mid |1 - |u_\varepsilon(x)|| \geq 1/2\}$ and $\{S_{i, \omega}\}_{i \in I_\omega}$ denote the collection of connected components of $\{x \in \omega \mid |u_\varepsilon(x)| \leq 1/2\}$ whose degree $d_{i, \omega} := \deg(u_\varepsilon/|u_\varepsilon|, \partial S_{i, \omega}) \neq 0$, we have*

$$\left\| \mu_{\varepsilon, \omega} - 2\pi \sum_{i \in I_\omega} d_{i, \omega} \delta_{a_{i, \omega}} \right\|_{C^{0,1}(\omega)^*} \leq C \max(r_\omega, \varepsilon) \left(1 + \int_\omega e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^2 + \int_{\partial\omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^1 \right),$$

where $a_{i, \omega}$ is the centroid of $S_{i, \omega}$, $r_\omega := \sum_{j \in J_\omega} \text{diam}(U_{j, \omega})$, and C is a universal constant.

In view of the previous corollary, it is important to bound from above r_ω , $d_{i, \omega}$, and $|I_\omega|$. Prior to doing so, let us recall the following result adapted from [Jer99].

Lemma 4.1. *Under the hypotheses of Corollary 4.1, there exists $\varepsilon_0(\gamma)$ such that, for any $\varepsilon < \varepsilon_0$ and for any face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, \delta))$ of a cube of the grid $\mathfrak{G}(b_\varepsilon, \delta)$, letting $\{S_{i, \omega}\}_{i \in I_\omega}$ be the collection of connected components of $\{x \in \omega \mid |u(x)| \leq 1/2\}$ whose degree $d_{i, \omega} \neq 0$, we have*

$$|d_{i, \omega}| \leq C \int_{S_{i, \omega}} |\nabla_{A_\varepsilon} u_\varepsilon|^2,$$

where C is a universal constant.

With the aid of the previous lemma we prove the following result.

Lemma 4.2. *Under the hypotheses of Corollary 4.1, there exists $\varepsilon_0(\gamma)$ such that, for any $\varepsilon < \varepsilon_0$ and for any face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, \delta))$ of a cube of the grid $\mathfrak{G}(b_\varepsilon, \delta)$, letting $\{S_{i, \omega}\}_{i \in I_\omega}$ be the collection of connected components of $\{x \in \omega \mid |u_\varepsilon(x)| \leq 1/2\}$ whose degree $d_{i, \omega} \neq 0$ and defining r_ω as in the corollary, we have*

$$(4.2) \quad \begin{aligned} |I_\omega| &\leq \sum_{i \in I_\omega} |d_{i, \omega}| \leq C \int_\omega e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^2, \\ r_\omega &\leq C\varepsilon \int_\omega e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^2, \end{aligned}$$

where C is a universal constant.

Proof. The first assertion immediately follows from Lemma 4.1. To prove (4.2) observe that, by the Cauchy-Schwarz inequality and the co-area formula, we have

$$\begin{aligned} \int_{\omega} e_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) d\mathcal{H}^2 &\geq \int_{\omega} |\nabla|u_{\varepsilon}||^2 + \frac{1}{\varepsilon^2}(1 - |u_{\varepsilon}|^2)^2 d\mathcal{H}^2 \\ &\geq \int_{\omega} \frac{|\nabla|u_{\varepsilon}|| (1 - |u_{\varepsilon}|^2)}{\varepsilon} d\mathcal{H}^2 \\ &= \int_{t=0}^{\infty} \frac{(1 - t^2)}{\varepsilon} \mathcal{H}^1(\{x \in \omega \mid |u_{\varepsilon}(x)| = t\}) dt. \end{aligned}$$

We deduce that there exist $t_1 \in [1/2, 3/4]$ and $t_2 \in [5/4, 3/2]$ such that

$$\mathcal{H}^1(\{x \in \omega \mid |u_{\varepsilon}(x)| = t_i\}) \leq C\varepsilon \int_{\omega} e_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) d\mathcal{H}^2$$

for $i = 1, 2$. This implies (4.2). \square

Remark 4.1. By combining Lemma 4.2 with (2.1c), we obtain

$$(4.3) \quad \sum_{\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, \delta))} |I_{\omega}| \leq C \int_{\mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, \delta))} e_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) d\mathcal{H}^2 \leq C\delta^{-1} F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}),$$

$$(4.4) \quad \sum_{\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, \delta))} \sum_{i \in I_{\omega}} |d_{i, \omega}| \leq C \int_{\mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, \delta))} e_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) d\mathcal{H}^2 \leq C\delta^{-1} F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}),$$

$$(4.5) \quad r_{\mathfrak{G}} := \sum_{\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, \delta))} r_{\omega} \leq C\varepsilon \int_{\mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, \delta))} e_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) d\mathcal{H}^2 \leq C\varepsilon\delta^{-1} F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}),$$

where $\sum_{\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, \delta))}$ denotes the sum over all the faces ω of cubes of the grid $\mathfrak{G}(b_{\varepsilon}, \delta)$.

5. 3D VORTEX APPROXIMATION CONSTRUCTION

In this section we construct a new polyhedral approximation of the vorticity $\mu(u_{\varepsilon}, A_{\varepsilon})$ of a configuration $(u_{\varepsilon}, A_{\varepsilon}) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ such that $F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) \leq \varepsilon^{-\gamma}$ for some $\gamma \in (0, 1)$. The notion of minimal connection, first introduced in [BCL86], plays a key role in our construction. We begin this section by reviewing this concept. We then define the function ζ and the function ζ for $d_{\partial\Omega}$, and describe how to smoothly approximate these functions. Lastly, we provide our 3D vortex approximation construction.

5.1. Minimal connections. Consider a collection $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$ of $2k$ points, where the p_i 's are the (not necessarily distinct) positive points and the n_i 's are the (not necessarily distinct) negative points. We define the length of a minimal connection joining the p_i 's to the n_i 's by

$$(5.1) \quad L(\mathcal{A}) := \min_{\sigma \in \mathfrak{S}_k} \sum_{i=1}^k |p_i - n_{\sigma(i)}|,$$

where \mathfrak{S}_k is the set of permutations of k indices and hereafter $|\cdot|$ denotes the Euclidean distance in \mathbb{R}^3 . We also define the 1-current $\mathbb{L}(\mathcal{A})$, a minimal connection associated to \mathcal{A} , as the sum in the sense of currents of the segments joining p_i to $n_{\sigma(i)}$, where $\sigma \in \mathfrak{S}_k$ is a permutation achieving the minimum in (5.1). Although there can be several minimal connections associated to a collection \mathcal{A} , we will make an arbitrary choice of one.

Let us now consider the distance

$$d_{\partial\Omega}(x_1, x_2) := \min\{|x_1 - x_2|, d(x_1, \partial\Omega) + d(x_2, \partial\Omega)\} \quad x_1, x_2 \in \mathbb{R}^3.$$

We define the length of a minimal connection joining the p_i 's to the n_i 's through $\partial\Omega$ by

$$(5.2) \quad L_{\partial\Omega}(\mathcal{A}) = \min_{\sigma \in \mathfrak{S}_k} \sum_{i=1}^k d_{\partial\Omega}(p_i, n_{\sigma(i)}).$$

In this case we define the 1-current $\mathbb{L}_{\partial\Omega}(\mathcal{A})$, a minimal connection through $\partial\Omega$ associated to \mathcal{A} , as the sum in the sense of currents of the segments joining p_i to $n_{\sigma(i)}$ when $d_{\partial\Omega}(p_i, n_{\sigma(i)}) = |p_i - n_{\sigma(i)}|$ and the (properly oriented) segments joining $p_i, n_{\sigma(i)}$ to $\partial\Omega$ when $d_{\partial\Omega}(p_i, n_{\sigma(i)}) = d(p_i, \partial\Omega) + d(n_{\sigma(i)}, \partial\Omega)$, where $\sigma \in \mathfrak{S}_k$ is a permutation achieving the minimum in (5.2). Once again, if the minimal connection is not unique we make an arbitrary choice of one.

5.1.1. *The function ζ .* The following lemma is a particular case of a well-known result proved in [BCL86].

Lemma 5.1. *Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$ be a configuration of positive and negative points. Assume, relabeling the points if necessary, that $L(\mathcal{A}) = \sum_{i=1}^k |p_i - n_i|$. Then there exists a 1-Lipschitz function $\zeta^* : \cup_{i=1, \dots, k} \{p_i, n_i\} \rightarrow \mathbb{R}$ such that*

$$L(\mathcal{A}) = \sum_{i=1}^k \zeta^*(p_i) - \zeta^*(n_i) \quad \text{and} \quad \zeta^*(n_i) = \zeta^*(p_i) - |p_i - n_i|.$$

Definition 5.1 (The function ζ). *Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$ be a configuration of positive and negative points. Denote by ζ^* the 1-Lipschitz function given by Lemma 5.1. We define the function $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$ via the formula*

$$\zeta(x) := \max_{i \in \{1, \dots, k\}} \left(\zeta^*(p_i) - \max_{j \in \{1, \dots, 2k\}} d_{(i,j)}(x) \right),$$

with

$$d_{(i,j)}(x) := \langle p_i - x, \nu_{(i,j)} \rangle, \quad \nu_{(i,j)} := \begin{cases} \frac{p_i - a_j}{|p_i - a_j|} & \text{if } p_i \neq a_j \\ 0 & \text{if } p_i = a_j \end{cases},$$

where here and in the rest of the paper the points a_i are defined as follows: if $j \in \{1, \dots, k\}$ then $a_j = p_j$, if $j \in \{k+1, \dots, 2k\}$ then $a_j = n_{j-k}$.

Lemma 5.2. *Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$ be a configuration of positive and negative points. Denote by $\zeta^* : \cup_{i=1, \dots, k} \{p_i, n_i\} \rightarrow \mathbb{R}$ the function given by Lemma 5.1 and define $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$ as in Definition 5.1. Then ζ is a 1-Lipschitz extension of ζ^* to \mathbb{R}^3 .*

Proof. It is easy to see that ζ is a 1-Lipschitz function. Let us check that

$$\zeta(p_i) = \zeta^*(p_i) \quad \text{and} \quad \zeta(n_i) = \zeta^*(n_i)$$

for every $i \in \{1, \dots, k\}$. Observe that

$$|d_{(i,j)}(x)| = |\langle p_i - x, \nu_{(i,j)} \rangle| \leq |p_i - x|.$$

Moreover

$$d_{(i,j)}(a_l) = |p_i - a_l| \quad \text{for any } a_l \in \mathcal{A},$$

thus

$$\zeta(a_l) = \max_{i \in \{1, \dots, k\}} (\zeta^*(p_i) - |p_i - a_l|).$$

Since ζ^* is 1-Lipschitz, we deduce that $\zeta(a_l) \leq \zeta^*(a_l)$. It follows that $\zeta(p_l) = \zeta^*(p_l)$ for every $l \in \{1, \dots, k\}$. We conclude the proof by noting that, for any $l \in \{1, \dots, k\}$,

$$\zeta(n_l) \geq \zeta^*(p_l) - |p_l - n_l| = \zeta^*(n_l).$$

□

Let us remark that this extension is not the same that appears in [BCL86]. As pointed out in the introduction (see Section 1.5), our strategy of proof of the main results combines the use of the co-area formula and the ball construction method applied on the level sets of the function ζ . In Section 3, we saw that in order to apply the ball construction on a surface we need to control its second fundamental form, but since ζ is only Lipschitz, we have no control on the second fundamental form of its level sets. For this reason we need to smoothly approximate this function and, moreover, to provide a quantitative estimate of the second fundamental form of the approximation. We have the following technical result, whose proof is postponed to Appendix A.

Proposition 5.1 (Quantitative smooth approximation of the function ζ). *Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$ be a configuration of positive and negative points. Assume, relabeling the points if necessary, that $L(\mathcal{A}) = \sum_{i=1}^k |p_i - n_i|$. Define $D_{\mathcal{A}} := \max_{a_i, a_j \in \mathcal{A}} |a_i - a_j|$ to be the maximum Euclidean distance between any of the points of \mathcal{A} . Then there exist universal constants $C, C_0, C_1 > 0$ such that, for any $\rho \in (0, 1/2)$ and for any $0 < \lambda \leq \lambda_0(\rho) = (C_0(2k)^{-5})^{1/\rho}$, there exists a smooth function $\zeta_\lambda : \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfying:*

- (1) $|L(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p_i) - \zeta_\lambda(n_i)| \leq CD_{\mathcal{A}}(2k)^6 \lambda^\rho$.
- (2) $\|\nabla \zeta_\lambda\|_{L^\infty(\mathbb{R}^3)} \leq 1$.

(3) For any $0 < \kappa < \lambda^{2\rho}/3$, there exists a set $T_\kappa \subset \mathbb{R}$ with

$$|T_\kappa| \leq C(2k)^8 \frac{\lambda}{\lambda^{2\rho} - 3\kappa},$$

such that, for any $t \in \mathbb{R} \setminus T_\kappa$, $\{x \mid \zeta_\lambda(x) = t\}$ is a complete submanifold of \mathbb{R}^3 whose second fundamental form is bounded by $C_1(\lambda^2\kappa)^{-1}$.

5.1.2. *The function ζ for $d_{\partial\Omega}$.* When the Euclidean distance is replaced with the distance through $\partial\Omega$ the following lemma can be proved (see [BCL86]).

Lemma 5.3. *Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\} \subset \Omega$ be a configuration of positive and negative points. Assume, relabeling the points if necessary, that $L_{\partial\Omega}(\mathcal{A}) = \sum_{i=1}^k d_{\partial\Omega}(p_i, n_i)$. Then there exists a function $\zeta^* : \cup_{i=1, \dots, k} \{p_i, n_i\} \rightarrow \mathbb{R}$, 1-Lipschitz for the distance $d_{\partial\Omega}$, such that*

$$L_{\partial\Omega}(\mathcal{A}) = \sum_{i=1}^k \zeta^*(p_i) - \zeta^*(n_i) \quad \text{and} \quad \zeta^*(n_i) = \zeta^*(p_i) - d_{\partial\Omega}(p_i, n_i).$$

Definition 5.2 (The function ζ for $d_{\partial\Omega}$). *Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$ be a configuration of positive and negative points. Denote by ζ^* the function given by Lemma 5.1. We define the function $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$ for $d_{\partial\Omega}$ via the formula*

$$\zeta(x) := \max_{i \in \{1, \dots, k\}} (\zeta^*(p_i) - d_i(x, \partial\Omega)),$$

where

$$d_i(x, \partial\Omega) := \min \left[\max \left(\max_{j \in \{1, \dots, 2k\}} d_{(i,j)}(x), d(p_i, \partial\Omega) - d(x, \partial\Omega) \right), d(p_i, \partial\Omega) + d(x, \partial\Omega) \right],$$

with

$$d_{(i,j)}(x) = \langle p_i - x, \nu_{(i,j)} \rangle, \quad \nu_{(i,j)} = \begin{cases} \frac{p_i - a_j}{|p_i - a_j|} & \text{if } p_i \neq a_j \\ 0 & \text{if } p_i = a_j \end{cases}.$$

Lemma 5.4. *Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$ be a configuration of positive and negative points. Denote by $\zeta^* : \cup_{i=1, \dots, k} \{p_i, n_i\} \rightarrow \mathbb{R}$ the function given by Lemma 5.3 and define $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$ as in Definition 5.2. Then ζ is a 1-Lipschitz extension of ζ^* to \mathbb{R}^3 , which is constant on $\partial\Omega$.*

Proof. It is easy to see that ζ is a 1-Lipschitz function. Let us check that

$$\zeta(p_i) = \zeta^*(p_i) \quad \text{and} \quad \zeta(n_i) = \zeta^*(n_i)$$

for every $i \in \{1, \dots, k\}$. By the proof of Lemma 5.2, we know that

$$d_{(i,j)}(a_l) = |p_i - a_l| \quad \text{for any } a_l \in \mathcal{A}.$$

By the triangular inequality, we deduce that

$$\max(|p_i - a_l|, d(p_i, \partial\Omega) - d(a_l, \partial\Omega)) = |p_i - a_l|.$$

Then

$$d_i(a_l, \partial\Omega) = \min(|p_i - a_l|, d(p_i, \partial\Omega) - d(a_l, \partial\Omega)) = d_{\partial\Omega}(p_i, a_l),$$

which implies that

$$\zeta(a_l) = \max_{i \in \{1, \dots, k\}} (\zeta^*(p_i) - d_{\partial\Omega}(p_i, a_l)).$$

Since ζ^* is 1-Lipschitz for the distance $d_{\partial\Omega}$, we have that $\zeta(a_l) \leq \zeta^*(a_l)$. It follows that $\zeta(p_l) = \zeta^*(p_l)$ for every $l \in \{1, \dots, k\}$, but

$$\zeta(n_l) \geq \zeta^*(p_l) - d_{\partial\Omega}(p_l, n_l) = \zeta^*(n_l).$$

Finally, observe that, for all $x \in \partial\Omega$,

$$d_i(x, \partial\Omega) := \min \left[\max \left(\max_{j \in \{1, \dots, 2k\}} d_{(i,j)}(x), d(p_i, \partial\Omega) \right), d(p_i, \partial\Omega) \right] = d(p_i, \partial\Omega).$$

Thus

$$\zeta(x) = \max_{i \in \{1, \dots, k\}} (\zeta^*(p_i) - d(p_i, \partial\Omega))$$

for all $x \in \partial\Omega$. □

We remark that this extension is not the same that appears in [BCL86]. In order to provide a lower bound for the free energy close to the boundary, we need to smoothly approximate the function ζ for $d_{\partial\Omega}$ and provide a quantitative estimate of the second fundamental form of the approximation. In this paper we describe two methods of doing this, which may be of independent interest. The first method is based on an analysis of the curvature of the boundary of the domain, which requires it to be of class C^2 . The second method is based on a polyhedral approximation of $\partial\Omega$, which in addition requires it to have strictly positive Gauss curvature.

Proposition 5.2. *[Quantitative smooth approximation of the function ζ for $d_{\partial\Omega}$ – First method] Assume that $\partial\Omega$ is of class C^2 . Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\} \subset \Omega$ be a configuration of positive and negative points. Assume, relabeling the points if necessary, that $L_{\partial\Omega}(\mathcal{A}) = \sum_{i=1}^k d_{\partial\Omega}(p_i, n_i)$. Then there exist constants θ_0, C, C_0, C_1 that depend only on $\partial\Omega$ such that, for any $\rho \in (0, 1/4)$ and for any $0 < \lambda < \lambda_0(\rho) = (C_0(2k)^{-5})^{1/\rho}$, there exists a smooth function $\zeta_\lambda : \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfying:*

$$(1) \quad |L_{\partial\Omega}(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p_i) - \zeta_\lambda(n_i)| \leq C(2k)^6 \lambda^\rho.$$

(2) Letting

$$(5.3) \quad \Omega_\lambda := \{x \in \Omega \mid 2\lambda^\rho < \text{dist}(x, \partial\Omega) < \theta_0 - 2\lambda^\rho\},$$

we have $\|\nabla \zeta_\lambda\|_{L^\infty(\Omega_\lambda)} \leq 1$.

$$(3) \quad |\zeta_\lambda(\{x \in \Omega \mid \text{dist}(x, \partial\Omega) \leq 2\lambda^\rho\})| \leq C\lambda^\rho.$$

(4) For any $0 < \kappa < \lambda^{2\rho}/3$, there exists a set $T_\kappa \subset \mathbb{R}$ with

$$|T_\kappa| \leq C \left((2k)^8 \frac{\lambda}{(\lambda^{2\rho} - 3\kappa)} + (2k)^6 \frac{\lambda}{(\lambda^{2\rho} - 3\kappa)^4} + (2k)^4 \lambda^{3\rho/4} \right),$$

such that, for any $t \in \zeta_\lambda(\Omega_\lambda) \setminus T_\kappa$, $\{x \mid \zeta_\lambda(x) = t\}$ is a complete submanifold of \mathbb{R}^3 whose second fundamental form is bounded by $C_1(\lambda^2\kappa)^{-1}$.

The proof of this proposition is deferred to Appendix B.

Proposition 5.3. [Quantitative smooth approximation of the function ζ for $d_{\partial\Omega}$ – Second method] Assume that $\partial\Omega$ is of class C^2 and has strictly positive Gauss curvature. Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\} \subset \Omega$ be a configuration of positive and negative points. Assume, relabeling the points if necessary, that $L_{\partial\Omega}(\mathcal{A}) = \sum_{i=1}^k d_{\partial\Omega}(p_i, n_i)$. Then there exist constants τ_0, C, C_0, C_1 , that depend only on $\partial\Omega$, such that, for any $\tau < \tau_0$, for any $\rho \in (0, 1/2)$, and for any

$$0 < \lambda < \lambda_0(\rho, \tau) = (C_0 \min \{(2k)^{-5}, (2k)^{-3}\tau^2, (2k)^{-1}\tau^4, \tau^5\})^{1/\rho},$$

there exists a smooth function $\zeta_\lambda : \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfying:

- (1) $|L_{\partial\Omega}(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p_i) - \zeta_\lambda(n_i)| \leq C(((2k)^6 + (2k)^4\tau^{-2} + (2k)^2\tau^{-4})\lambda^\rho + 2k\tau^2)$.
- (2) Letting

$$\Omega_\lambda := \{x \in \Omega \mid 2\lambda < \text{dist}(x, \partial\Omega)\},$$

we have $\|\nabla\zeta_\lambda\|_{L^\infty(\Omega_\lambda)} \leq 1$.

- (3) $|\zeta_\lambda(\Omega \setminus \Omega_\lambda)| \leq C(\tau^2 + \lambda)$.
- (4) For any $0 < \kappa < \lambda^{2\rho}/3$, there exists a set $T_\kappa \subset \mathbb{R}$ with

$$|T_\kappa| \leq C((2k)^8 + \tau^{-8}) \frac{\lambda}{(\lambda^{2\rho} - 3\kappa)},$$

such that, for any $t \in \zeta_\lambda(\Omega_\lambda) \setminus T_\kappa$, $\{x \mid \zeta_\lambda(x) = t\}$ is a complete submanifold of \mathbb{R}^3 whose second fundamental form is bounded by $C_1(\lambda^2\kappa)^{-1}$.

We point out that in this proposition the parameter τ is associated to the polyhedral approximation of $\partial\Omega$. The proof of this result is deferred to Appendix C.

5.2. Construction of the vorticity approximation. Let $\gamma \in (0, 1)$ and consider a configuration $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq \varepsilon^{-\gamma}$. Then Lemma 2.1 provides a grid $\mathfrak{G}(b_\varepsilon, \delta)$ satisfying (2.1). We begin by constructing our approximation in the cubes of the grid. For each cube $\mathcal{C}_l \in \mathfrak{G}(b_\varepsilon, \delta)$, Corollary 4.1 gives the existence of points $a_{i,\omega}$ and integers $d_{i,\omega} \neq 0$ such that

$$\mu_{\varepsilon,\omega} \approx 2\pi \sum_{i \in I_\omega} d_{i,\omega} \delta_{a_{i,\omega}},$$

for each of the six faces $\omega \subset \partial\mathcal{C}_l$ of the cube \mathcal{C}_l . Observe that, since $\partial\mu(u_\varepsilon, A_\varepsilon) = 0$ relative to \mathcal{C}_l , we have

$$\sum_{\omega \subset \partial\mathcal{C}_l} \sum_{i \in I_\omega} d_{i,\omega} = 0.$$

Then, we define a configuration $\mathcal{A}_l := \{p_1, \dots, p_{k_l}, n_1, \dots, n_{k_l}\}$ of positive and negative points associated to $\partial\mathcal{C}_l$, by repeating the points $a_{i,\omega}$ according to their degree

$d_{i,\omega}$, for each of the six faces ω of the cube \mathcal{C}_l . The previous observation implies that the number of positive points p_i 's and negative points n_i 's of the collection \mathcal{A}_l are equal. We note that

$$2k_l = \sum_{\omega \subset \partial \mathcal{C}_l} \sum_{i \in I_\omega} |d_{i,\omega}|.$$

Consider the minimal connection $\mathbb{L}(\mathcal{A}_l)$ associated to \mathcal{A}_l . It may happen that the segment connecting some p_i to $n_{\sigma(i)}$ in $\mathbb{L}(\mathcal{A}_l)$ belongs to one of the faces ω of the cube \mathcal{C}_l . In this case we define a new connection $\tilde{\mathbb{L}}(\mathcal{A}_l)$ by replacing the original segment connecting p_i to $n_{\sigma(i)}$ with a Lipschitz curve connecting p_i to $n_{\sigma(i)}$ from the inside (preserving the orientation), so that its intersection with $\partial \mathcal{C}_l$ is given by $\{p_i, n_{\sigma(i)}\}$. This process can be performed in such a way that $|L(\mathcal{A}_l) - \tilde{\mathbb{L}}(\mathcal{A}_l)|$ is less than an arbitrarily small number. We remark that the resulting connection $\tilde{\mathbb{L}}(\mathcal{A}_l)$ is a polyhedral 1-current whose intersection with $\partial \mathcal{C}_l$ is equal to $\cup_{i=1, \dots, k_l} \{p_i, n_i\}$. We define

$$\nu_{\varepsilon, \mathcal{C}_l} := 2\pi \tilde{\mathbb{L}}(\mathcal{A}_l) \quad \text{in } \mathcal{C}_l$$

for every cube $\mathcal{C}_l \in \mathfrak{G}(b_\varepsilon, \delta)$.

We now construct our vorticity approximation in $\bar{\Theta}$ (recall (2.2)). Once again Corollary 4.1 gives the existence of points $a_{i,\omega}$ and integers $d_{i,\omega} \neq 0$ such that

$$\mu_{\varepsilon, \omega} \approx 2\pi \sum_{i \in I_\omega} d_{i,\omega} \delta_{a_{i,\omega}}$$

for each face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, \delta))$ of a cube of the grid such that $\omega \subset \partial \mathfrak{G}$. Then, we define a configuration $\mathcal{A}_{\partial \mathfrak{G}} := \{p_1, \dots, p_{k_{\partial \mathfrak{G}}}, n_1, \dots, n_{k_{\partial \mathfrak{G}}}\}$ of positive and negative points associated to $\partial \mathfrak{G}$ by repeating the points $a_{i,\omega}$ according to their degree $d_{i,\omega}$, for each face $\omega \in \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, \delta))$ of a cube of the grid such that $\omega \subset \partial \mathfrak{G}$. Observe that, since $\partial \mu(u_\varepsilon, A_\varepsilon) = 0$ relative to $\partial \mathfrak{G}$, we have

$$\sum_{\omega \subset \partial \mathfrak{G}} \sum_{i \in I_\omega} d_{i,\omega} = 0,$$

which ensures that the number of positive points p_i 's and negative points n_i 's of the collection $\mathcal{A}_{\partial \Omega}$ are equal. We note that

$$2k_{\partial \mathfrak{G}} = \sum_{\omega \subset \partial \mathfrak{G}} \sum_{i \in I_\omega} |d_{i,\omega}|.$$

One might want to define the vorticity approximation close to the boundary as the minimal connection $\mathbb{L}_{\partial \Omega}(\mathcal{A}_{\partial \mathfrak{G}})$ through $\partial \Omega$ associated to $\mathcal{A}_{\partial \mathfrak{G}}$. Unfortunately we cannot do this, because it is not possible to rule out the possibility of having

$$\mathbb{L}_{\partial \Omega}(\mathcal{A}_{\partial \mathfrak{G}}) \cap (\bar{\Omega} \setminus \bar{\Theta}) \neq \emptyset.$$

For this reason, we consider the distance

$$\mathfrak{d}_{\partial \Omega}(x_1, x_2) := \min\{\mathfrak{d}(x_1, x_2), d(x_1, \partial \Omega) + d(x_2, \partial \Omega)\} \quad x_1, x_2 \in \partial \mathfrak{G},$$

where \mathfrak{d} denotes the geodesic distance on $\partial\mathfrak{G}$.

Let us define

$$(5.4) \quad L_{\mathfrak{d}_{\partial\Omega}}(\mathcal{A}_{\partial\mathfrak{G}}) = \min_{\sigma \in \mathfrak{S}_{k_{\partial\mathfrak{G}}}} \sum_{i=1}^{k_{\partial\mathfrak{G}}} \mathfrak{d}_{\partial\Omega}(p_i, n_{\sigma(i)})$$

and the 1-current $\mathbb{L}_{\mathfrak{d}_{\partial\Omega}}(\mathcal{A}_{\partial\mathfrak{G}})$, a minimal connection through $\partial\Omega$ associated to $\mathcal{A}_{\partial\mathfrak{G}}$ contained in $\bar{\Theta}$, as the sum in the sense of currents of the geodesics (on $\partial\mathfrak{G}$) joining p_i to $n_{\sigma(i)}$ when $\mathfrak{d}_{\partial\Omega}(p_i, n_{\sigma(i)}) = \mathfrak{d}(p_i, n_{\sigma(i)})$ and the (properly oriented) segments joining $p_i, n_{\sigma(i)}$ to $\partial\Omega$ when $d_{\partial\Omega}(p_i, n_{\sigma(i)}) = d(p_i, \partial\Omega) + d(n_{\sigma(i)}, \partial\Omega)$, where $\sigma \in \mathfrak{S}_{k_{\partial\mathfrak{G}}}$ is a permutation achieving the minimum in (5.4). If the minimal connection is not unique we make an arbitrary choice of one.

Performing a replacement argument (from the inside) in $\bar{\Theta}$, analogous to the one described above, we define a new connection $\tilde{\mathbb{L}}_{\mathfrak{d}_{\partial\Omega}}(\mathcal{A}_{\partial\mathfrak{G}})$, with $|L_{\mathfrak{d}_{\partial\Omega}}(\mathcal{A}_{\partial\mathfrak{G}}) - \tilde{\mathbb{L}}_{\mathfrak{d}_{\partial\Omega}}(\mathcal{A}_{\partial\mathfrak{G}})|$ less than an arbitrarily small number, whose intersection with $\partial\mathfrak{G}$ is equal to $\cup_{i=1, \dots, k_{\partial\mathfrak{G}}} \{p_i, n_i\}$ and which is contained in $\bar{\Theta}$. We set

$$\nu_{\varepsilon, \Theta} := 2\pi \tilde{\mathbb{L}}_{\mathfrak{d}_{\partial\Omega}}(\mathcal{A}_{\partial\mathfrak{G}}) \quad \text{in } \bar{\Theta}.$$

Finally, we define our polyhedral approximation ν_ε of the vorticity $\mu(u_\varepsilon, A_\varepsilon)$ by

$$(5.5) \quad \nu_\varepsilon := \sum_{\mathcal{C}_i \in \mathfrak{C}(b_\varepsilon, \delta)} \nu_{\varepsilon, \mathcal{C}_i} + \nu_{\varepsilon, \Theta},$$

where the sums are understood in the sense of currents.

We observe that the topological degree depends on the orientation of the domain in which it is computed. If a face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, \delta))$ belongs to two cubes C_1 and C_2 of the grid, then its associated collection of degrees $d_{i, \omega}$'s for C_1 is equal to minus its associated collection of degrees for C_2 . Of course the same occurs for those faces ω belonging to one of the cubes of the grid and to $\partial\mathfrak{G}$.

On the other hand (2.1a) implies that, for any face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, \delta))$, the intersection between the collection of points $a_{i, \omega}$'s and $\mathfrak{R}_1(\mathfrak{G}(b_\varepsilon, \delta))$ is empty.

By combining these arguments we conclude that the 1-currents $\nu_{\mathcal{C}_i}$'s and ν_Θ have a good compatibility condition between each other. Hence, by construction, ν_ε is a polyhedral 1-current such that $\partial\nu_\varepsilon = 0$ relative to Ω . In addition it approximates well $\mu(u_\varepsilon, A_\varepsilon)$ in an appropriate norm, as we shall show in Section 8.

5.2.1. *An important remark towards the proof of the lower bound close to the boundary.* Let us study the 1-current $\mathbb{L}_{\mathfrak{d}_{\partial\Omega}}(\mathcal{A}_{\partial\mathfrak{G}})$ defined above. We are interested in the situation $\mathfrak{d}_{\partial\Omega}(p_i, n_{\sigma(i)}) = \mathfrak{d}(p_i, n_{\sigma(i)})$, where σ denotes a (fixed) permutation achieving the minimum in (5.4). Since $d(x, \partial\Omega) \leq 2\delta$ for any $x \in \partial\mathfrak{G}$, we have two possibilities:

- $\mathfrak{d}(p_i, n_{\sigma(i)}) = |p_i - n_{\sigma(i)}|$ and there exists a face $\omega \subset \partial\mathfrak{G}$ which contains both points.

- $\mathfrak{d}(p_i, n_{\sigma(i)}) \neq |p_i - n_{\sigma(i)}|$ and there exist (different) faces $w_1, \dots, w_{J(i)} \subset \partial\mathfrak{G}$, with $2 \geq J(i) < C^*$ for some constant $C^* > 0$ that depends only on the boundary, such that $p_i \in \omega_1$, $n_{\sigma(i)} \in \omega_{J(i)}$, and

$$\mathfrak{d}(p_i, n_{\sigma(i)}) = |p_i - q_1| + |q_1 - q_2| + \dots + |q_{J(i)-2} - q_{J(i)-1}| + |q_{J(i)-1} - n_{\sigma(i)}|,$$

where the points q_j are such that $q_j \in \partial\omega_j \cap \partial\omega_{j+1}$, $j = 1, \dots, J(i) - 1$.

If the second situation occurs for some i , we will enlarge the collection of points $\mathcal{A}_{\partial\mathfrak{G}}$. We proceed as follows: for any i for which the second situation happens, we add, for $j = 1, \dots, J(i) - 1$, the point q_j to the collection twice; both as a positive and negative point (i.e. with degree $+1$ and -1). This yields a new collection

$$(5.6) \quad \tilde{\mathcal{A}}_{\partial\mathfrak{G}} = \{p_1, \dots, p_{\tilde{k}_{\partial\mathfrak{G}}}, n_1, \dots, n_{\tilde{k}_{\partial\mathfrak{G}}}\}$$

of positive and negative points, which contains $\mathcal{A}_{\partial\mathfrak{G}}$. In particular, we have that

$$(5.7) \quad \tilde{k}_{\partial\mathfrak{G}} \leq C^* k_{\partial\mathfrak{G}} \leq C^* \sum_{\omega \subset \partial\mathfrak{G}} \sum_{i \in I_\omega} |d_{i,\omega}|.$$

Moreover,

$$(5.8) \quad L_{\partial\Omega}(\tilde{\mathcal{A}}_{\partial\mathfrak{G}}) := \min_{\sigma \in \tilde{\mathfrak{S}}_{\tilde{k}_{\partial\mathfrak{G}}}} \sum_{i=1}^{\tilde{k}_{\partial\mathfrak{G}}} \mathfrak{d}_{\partial\Omega}(p_i, n_{\sigma(i)}) = L_{\partial\Omega}(\mathcal{A}_{\partial\mathfrak{G}}).$$

The commodity of using this new collection is that there exists a permutation $\sigma^* \in \tilde{\mathfrak{S}}_{\tilde{k}_{\partial\mathfrak{G}}}$ achieving the minimum in (5.8), which is naturally derived from the previous construction, such that if $\mathfrak{d}_{\partial\Omega}(p_i, n_{\sigma^*(i)}) = \mathfrak{d}(p_i, n_{\sigma^*(i)})$ then $\mathfrak{d}(p_i, n_{\sigma^*(i)}) = |p_i - n_{\sigma^*(i)}|$ and there exists a face $\omega \subset \partial\mathfrak{G}$ which contains both points. This in particular implies that

$$\mathfrak{d}_{\partial\Omega}(p_i, n_{\sigma^*(i)}) = \min\{\mathfrak{d}(p_i, n_{\sigma^*(i)}), d(p_i, \partial\Omega) + d(n_{\sigma^*(i)}, \partial\Omega)\} = d_{\partial\Omega}(p_i, n_{\sigma^*(i)}).$$

Finally, by [San01, Lemma 2,2], which is a slight modification of the previously cited well-known result in [BCL86], there exists a 1-Lipschitz function $\zeta^* : \cup_{i=1, \dots, \tilde{k}_{\partial\mathfrak{G}}} \{p_i, n_{\sigma^*(i)}\} \rightarrow \mathbb{R}$ such that

$$(5.9) \quad L_{\partial\Omega}(\tilde{\mathcal{A}}_{\partial\mathfrak{G}}) = \sum_{i=1}^{\tilde{k}_{\partial\mathfrak{G}}} \zeta^*(p_i) - \zeta^*(n_{\sigma^*(i)}) \quad \text{and} \quad \zeta^*(p_i) - \zeta^*(n_{\sigma^*(i)}) = \mathfrak{d}_{\partial\Omega}(p_i, n_{\sigma^*(i)}).$$

Combining this with our previous observation, we get

$$(5.10) \quad \zeta^*(p_i) - \zeta^*(n_{\sigma^*(i)}) = d_{\partial\Omega}(p_i, n_{\sigma^*(i)}).$$

In particular, we can extend this function by defining the function ζ for $d_{\partial\Omega}$ as in Definition 5.2, and therefore in the proof of the lower bound close to the boundary (see Section 7) will be enough to consider a quantitative smooth approximation of this extension. It is worth remarking that in this case we cannot ensure that $L_{\partial\Omega}(\tilde{\mathcal{A}}_{\partial\mathfrak{G}}) = L_{\partial\Omega}(\mathcal{A}_{\partial\mathfrak{G}})$, but since (5.10) holds, arguing almost as readily as in the proofs of Propositions 5.2 and 5.3, one can show that there exists a smooth function

ζ_λ associated to $\tilde{\mathcal{A}}_{\partial\mathfrak{G}}$ satisfying these propositions with the quantity $L_{\partial\Omega}(\tilde{\mathcal{A}}_{\partial\mathfrak{G}})$ being replaced by $\sum_{i=1}^{\tilde{k}_{\partial\mathfrak{G}}} \zeta^*(p_i) - \zeta^*(n_{\sigma^*(i)})$. In particular, by combining this with (5.9) and (5.10), we conclude that

$$\left| L_{\partial\Omega}(\tilde{\mathcal{A}}_{\partial\mathfrak{G}}) - \sum_{i=1}^{\tilde{k}_{\partial\mathfrak{G}}} \zeta_\lambda(p_i) - \zeta_\lambda(n_{\sigma^*(i)}) \right| \leq \text{small (quantitative) error.}$$

The function ζ_λ that we use in Section 7 is precisely the function described here.

5.2.2. *The support of ν_ε .* To end this section we present a result about the support of ν_ε .

Lemma 5.5. *Let $\gamma \in (0, 1)$ and assume that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq \varepsilon^{-\gamma}$, so that, by Lemma 2.1, there exists a grid $\mathfrak{G}(b_\varepsilon, \delta)$ satisfying (2.1). For each face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, \delta))$ of a cube of the grid, let $|I_\omega|$ be the number of connected components of $\{x \in \omega \mid |u_\varepsilon(x)| \leq 1/2\}$ whose degree is different from zero. Then, letting*

$$(5.11) \quad \mathfrak{G}_0 := \{\mathcal{C}_l \in \mathfrak{G} \mid \sum_{\omega \subset \partial\mathcal{C}_l} |I_\omega| > 0\}$$

and defining ν_ε by (5.5), we have

$$\text{supp}(\nu_\varepsilon) \subset S_{\nu_\varepsilon} := \bigcup_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{C}_l \cup \begin{cases} \bar{\Theta} & \text{if } \sum_{\omega \subset \partial\mathfrak{G}} |I_\omega| > 0 \\ \emptyset & \text{if } \sum_{\omega \subset \partial\mathfrak{G}} |I_\omega| = 0 \end{cases}.$$

Moreover

$$|S_{\nu_\varepsilon}| \leq C\delta(1 + \delta F_\varepsilon(u_\varepsilon, A_\varepsilon)),$$

where C is a constant depending only on $\partial\Omega$.

Proof. The first assertion follows readily from the definition of ν_ε . Recall that, by (4.3), the number of faces $\omega \in \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, \delta))$ of a cube of the grid such that $|I_\omega| > 0$ is bounded above by $C\delta^{-1}F_\varepsilon(u_\varepsilon, A_\varepsilon)$. We deduce that $\#\{\mathcal{C}_l \mid \mathcal{C}_l \in \mathfrak{G}_0\}$ is bounded above by $C\delta^{-1}F_\varepsilon(u_\varepsilon, A_\varepsilon)$. By noting that $|\Theta| \leq C\delta$, for some constant C depending only on $\partial\Omega$, we conclude that

$$|S_{\nu_\varepsilon}| \leq \sum_{\mathcal{C}_l \in \mathfrak{G}_0} |\mathcal{C}_l| + |\Theta| \leq \delta^3 \#\{\mathcal{C}_l \mid \mathcal{C}_l \in \mathfrak{G}_0\} + C\delta \leq C\delta(1 + \delta F_\varepsilon(u_\varepsilon, A_\varepsilon)).$$

□

6. LOWER BOUND FOR $E_\varepsilon(u_\varepsilon)$ FAR FROM THE BOUNDARY

In this section we provide a lower bound, in the spirit of (1.1), for the energy without magnetic field $E_\varepsilon(u_\varepsilon)$ in the union of cubes of the grid $\mathfrak{G}(b_\varepsilon, \delta)$ given by Lemma 2.1. The proof relies on a slicing procedure based on the level sets of the smooth approximation of the function ζ constructed in Appendix A and on the ball construction method on a surface of Section 3.

Proposition 6.1. *Let $m, M > 0$ and assume that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) = M_\varepsilon \leq M |\log \varepsilon|^m$. For any $b, q > 0$, there exists $\varepsilon_0 > 0$ depending only on b, q, m , and M , such that, for any $\varepsilon < \varepsilon_0$, letting $\mathfrak{G}(b_\varepsilon, \delta)$ denote the grid given by Lemma 2.1 with $\delta = \delta(\varepsilon) = |\log \varepsilon|^{-q}$, and defining ν_ε by (5.5) and \mathfrak{G}_0 by (5.11), if*

$$(6.1) \quad E_\varepsilon(u_\varepsilon, \cup_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{C}_l) \leq KM_\varepsilon \quad \text{and} \quad \sum_{\mathcal{C}_l \in \mathfrak{G}_0} \int_{\partial \mathcal{C}_l} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2 \leq K\delta^{-1}M_\varepsilon$$

for some universal constant K , then

$$E_\varepsilon(u_\varepsilon, \cup_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{C}_l) \geq \frac{1}{2} \sum_{\mathcal{C}_l \in \mathfrak{G}_0} |\nu_{\varepsilon, \mathcal{C}_l}| \left(\log \frac{1}{\varepsilon} - \log C \frac{M_\varepsilon^{56} |\log \varepsilon|^{7(1+b)}}{\delta^{55}} \right) - \frac{C}{|\log \varepsilon|^b},$$

where C is a universal constant.

Proof. Let us first prove an estimate for each cube of the grid.

Step 1. Lower bound via the co-area formula. We consider a cube $\mathcal{C}_l \in \mathfrak{G}_0$. For each of the six faces ω of \mathcal{C}_l , denote by $\{S_{i,\omega}\}_{i \in I_\omega}$ the collection of connected components of $\{x \in \omega \mid |u_\varepsilon(x)| \leq 1/2\}$. We define

$$S_l := \cup_{\omega \subset \partial \mathcal{C}_l} \cup_{i \in I_\omega} S_{i,\omega}.$$

Note that $|u_\varepsilon(x)| > 1/2$ for any $x \in \partial \mathcal{C}_l \setminus S_l$.

Denote by $\mathcal{A}_l = \{p_1, \dots, p_{k_l}, n_1, \dots, n_{k_l}\}$ the configuration of positive and negative points associated to the cube \mathcal{C}_l (see Section 5.2). For parameters $\rho \in (0, 1/2)$, $0 < \lambda = \lambda(l) \leq (C_0(2k_l)^{-5})^{1/\rho}$, and $0 < \kappa = \kappa(\lambda) < \lambda^{2\rho}/3$ to be chosen later, let ζ_λ be the smooth function associated to \mathcal{A}_l by Proposition 5.1 and consider the set T_κ defined there. Here the constant C_0 is the universal constant appearing in the proposition.

Letting

$$\tilde{\mathcal{C}}_l := \{x \mid d(x, \mathcal{C}_l) < C_1 \lambda^2 \kappa, \text{ proj}_{\mathcal{C}_l} x \notin S_l\},$$

where C_1 is the universal constant appearing in the third statement of Proposition 5.1, we define $v_\varepsilon : \overline{\tilde{\mathcal{C}}_l} \rightarrow \mathbb{C}$ via the formula

$$v_\varepsilon(x) = u_\varepsilon(\text{proj}_{\mathcal{C}_l} x) \quad x \in \overline{\tilde{\mathcal{C}}_l}.$$

Observe that

$$E_\varepsilon(u_\varepsilon, \mathcal{C}_l) \geq E_\varepsilon(v_\varepsilon, \tilde{\mathcal{C}}_l) - C_1 \lambda^2 \kappa \int_{\partial \mathcal{C}_l \setminus S_l} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2.$$

In particular, if $\lambda^2 \kappa$ is small enough then $E_\varepsilon(v_\varepsilon, \tilde{\mathcal{C}}_l) \leq 2M_\varepsilon$. We also define

$$U_\lambda := \zeta_\lambda(\{x \in \partial \tilde{\mathcal{C}}_l \mid \text{proj}_{\mathcal{C}_l} x \in S_l\})$$

and note that

$$|U_\lambda| \leq \sum_{\omega \subset \partial \tilde{\mathcal{C}}_l} \sum_{i \in I_\omega} \text{diam}(S_{i,\omega}) + (2k_l)C_1\lambda^2\kappa.$$

Since $|\nabla \zeta_\lambda| \leq 1$, using the co-area formula, we deduce that

$$E_\varepsilon(v_\varepsilon, \tilde{\mathcal{C}}_l) \geq \int_{\tilde{\mathcal{C}}_l} e_\varepsilon(v_\varepsilon) |\nabla \zeta_\lambda| = \int_{t \in \mathbb{R}} \int_{\{\zeta_\lambda = t\} \cap \tilde{\mathcal{C}}_l} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 dt.$$

Step 2. Lower bound via the ball construction on a surface. We would now like to apply the results of Section 3. Let us consider a small number $\gamma > 0$ and define

$$V_\gamma := \left\{ t \in \mathbb{R} \mid \int_{\{\zeta_\lambda = t\} \cap \tilde{\mathcal{C}}_l} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 > \frac{1}{\gamma} M_\varepsilon \right\}.$$

Note that $|V_\gamma| \leq 2K\gamma$. Finally, let us define

$$T_{\text{bad}} := T_\kappa \cup U_\lambda \cup V_\gamma, \quad \Sigma_t := \{\zeta_\lambda = t\} \cap \tilde{\mathcal{C}}_l, \quad t_* := \min_{a_i \in \mathcal{A}_l} \zeta_\lambda(a_i), \quad \text{and} \quad t^* := \max_{a_i \in \mathcal{A}_l} \zeta_\lambda(a_i).$$

For $t \in T_{\text{good}} := [t_*, t^*] \setminus T_{\text{bad}}$ it holds that:

- $\int_{\Sigma_t} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 \leq \gamma^{-1} M_\varepsilon$.
- $\{\zeta_\lambda = t\}$ is a surface whose second fundamental form is bounded by $C_1(\lambda^2\kappa)^{-1}$.
Note that this surface is necessarily oriented since it is a level set of ζ_λ .
- $\partial \Sigma_t = \{\zeta_\lambda = t\} \cap \partial \tilde{\mathcal{C}}_l$.
- $|v_\varepsilon(x)| > 1/2$ if $d(x, \partial \Sigma_t) < C_1\lambda^2\kappa$.

Then Corollary 3.1 applied to v_ε on Σ_t with $Q_\varepsilon = C_1(\lambda^2\kappa)^{-1}$ and $M_\varepsilon = \gamma^{-1}M_\varepsilon$, yields that, for any $t \in T_{\text{good}}$,

$$\int_{\Sigma_t} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 \geq \pi |\deg(v_\varepsilon, \partial \Sigma_t)| \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right).¹$$

We point out that we cannot directly apply Corollary 3.1 to u_ε on $\Sigma = \{\zeta_\lambda = t\} \cap \tilde{\mathcal{C}}_l$ (for $t \in T_{\text{good}}$), because (3.5) does not necessarily hold in this case. For this reason, we extended the function u_ε to $\tilde{\mathcal{C}}_l$ in the previous step.

Noting that $\partial \Sigma_t = \partial(\{\zeta_\lambda \geq t\} \cap \partial \tilde{\mathcal{C}}_l)$, we deduce that

$$\deg(v_\varepsilon, \partial \Sigma_t) = d(t) := \#\{i \mid \zeta_\lambda(p_i) > t\} - \#\{i \mid \zeta_\lambda(n_i) > t\}.$$

¹To apply the corollary we actually need $\lambda^{-1}, \kappa^{-1}$, and γ^{-1} to be bounded above by positive powers of $|\log \varepsilon|$. For this reason, and because of our choice of these parameters in terms of $\delta(\varepsilon)$ (see Step 3), we require δ to be a negative power of $|\log \varepsilon|$ in the statement.

By combining our previous estimates, we find

$$\begin{aligned} E_\varepsilon(v_\varepsilon, \tilde{\mathcal{C}}_l) &\geq \int_{t \in T_{\text{good}}} \int_{\Sigma_t} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 dt \\ &\geq \pi \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right) \int_{t \in T_{\text{good}}} d(t) dt \\ &\geq \pi \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right) \left(\int_{t_*}^{t^*} d(t) dt - \int_{t \in T_{\text{bad}}} |d(t)| dt \right). \end{aligned}$$

Moreover, for any $t \in T_{\text{bad}}$,

$$|d(t)| = |\#\{i \mid \zeta_\lambda(p_i) > t\} - \#\{i \mid \zeta_\lambda(n_i) > t\}| \leq k_l.$$

Then

$$\int_{t \in T_{\text{bad}}} |d(t)| dt \leq k_l |T_{\text{bad}}| \leq k_l (|T_\kappa| + |U_\lambda| + |V_\gamma|).$$

On the other hand, observe that

$$\int_{t_*}^{t^*} d(t) dt = \int_{t_*}^{t^*} (\#\{i \mid \zeta_\lambda(p_i) > t\} - \#\{i \mid \zeta_\lambda(n_i) > t\}) dt = \sum_{i=1}^{k_\lambda} \zeta_\lambda(p_i) - \zeta_\lambda(n_i).$$

Since

$$\left| L(\mathcal{A}_l) - \sum_{i=1}^{k_l} \zeta_\lambda(p_i) - \zeta_\lambda(n_i) \right| \leq C D_{\mathcal{A}_l} (2k_l)^6 \lambda^\rho \leq C \delta (2k_l)^6 \lambda^\rho$$

and remembering that $|2\pi L(\mathcal{A}_l) - |\nu_{\varepsilon, \mathcal{C}_l}|$ can be taken arbitrarily small, we conclude that

$$\int_{t_*}^{t^*} d(t) dt \geq \frac{1}{2\pi} |\nu_{\varepsilon, \mathcal{C}_l}| - C (\delta (2k_l)^6 \lambda^\rho + (2k_l) \lambda),$$

where hereafter $C > 0$ denotes a universal constant that may change from line to line. Collecting our previous computations, we find

$$E_\varepsilon(u_\varepsilon, \mathcal{C}_l) \geq \frac{1}{2} |\nu_{\varepsilon, \mathcal{C}_l}| \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right) - \mathcal{E}_l,$$

where

$$\mathcal{E}_l := C (\delta (2k_l)^6 \lambda^\rho + k_l (|T_\kappa| + |U_\lambda| + |V_\gamma|)) \log \frac{1}{\varepsilon} + C_1 \lambda^2 \kappa \int_{\partial \mathcal{C}_l \setminus S_l} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2.$$

Step 3. Choice of the parameters. We now want to combine the estimates found for cubes in \mathfrak{G}_0 . Observe that if λ and κ are chosen independent of l then

$$E_\varepsilon(u_\varepsilon, \cup_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{C}_l) \geq \frac{1}{2} \sum_{\mathcal{C}_l \in \mathfrak{G}_0} |\nu_{\varepsilon, \mathcal{C}_l}| \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right) - \sum_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{E}_l.$$

Our objective is then to choose the parameters $\lambda, \kappa = \kappa(\lambda)$, and γ independent of l and such that $\sum_{\mathcal{C}_l \subset \mathfrak{G}_0} \mathcal{E}_l \leq C|\log \varepsilon|^{-b}$. Since (4.4) implies that

$$\sum_{\mathcal{C}_l \in \mathfrak{G}_0} 2k_l \leq C\delta^{-1}M_\varepsilon,$$

we can achieve our goal provided that λ satisfies $\lambda \leq (C_0(C\delta^{-1}M_\varepsilon)^{-5})^{1/\rho}$.

Letting $\kappa = \lambda^{2\rho}/6$, and using (4.4), (4.5), and (6.1), we are led to

$$\sum_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{E}_l \leq C \log \frac{1}{\varepsilon} \left(\frac{M_\varepsilon^6}{\delta^5} \lambda^\rho + \frac{M_\varepsilon^9}{\delta^9} \lambda^{1-2\rho} + \frac{M_\varepsilon^2}{\delta^2} \varepsilon + \frac{M_\varepsilon}{\delta} \gamma \right).$$

Then, choosing $\rho = 6/21$,

$$\lambda = \left(\frac{1}{|\log \varepsilon|^{1+b}} \frac{\delta^9}{M_\varepsilon^9} \right)^{\frac{21}{9}}, \quad \text{and} \quad \gamma = \frac{1}{|\log \varepsilon|^{1+b}} \frac{\delta}{M_\varepsilon},$$

we easily check that there exists $\varepsilon_0 > 0$ depending only on b, q, m , and M , such that $\sum_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{E}_l \leq C|\log \varepsilon|^{-b}$ for any $0 < \varepsilon < \varepsilon_0$. Thus

$$E_\varepsilon(u_\varepsilon, \cup_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{C}_l) \geq \frac{1}{2} \sum_{\mathcal{C}_l \in \mathfrak{G}_0} |\nu_{\varepsilon, \mathcal{C}_l}| \left(\log \frac{1}{\varepsilon} - \log C \frac{M_\varepsilon^{56} |\log \varepsilon|^{7(1+b)}}{\delta^{55}} \right) - \frac{C}{|\log \varepsilon|^b}.$$

The proposition is proved. \square

7. LOWER BOUND FOR $E_\varepsilon(u_\varepsilon)$ CLOSE TO THE BOUNDARY

In this section we provide a lower bound, in the spirit of (1.1), for the energy without magnetic field $E_\varepsilon(u_\varepsilon)$ in Θ . The proof relies on a slicing procedure based on the level sets of the smooth approximation of the function ζ for $d_{\partial\Omega}$ constructed in Appendix B and on the ball construction method on a surface of Section 3.

Proposition 7.1. *Suppose that $\partial\Omega$ is of class C^2 . Let $m, M > 0$ and assume that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) = M_\varepsilon \leq M|\log \varepsilon|^m$. For any $b, q > 0$, there exists $\varepsilon_0 > 0$ depending only on b, q, m, M , and $\partial\Omega$, such that, for any $\varepsilon < \varepsilon_0$, letting $\mathfrak{G}(b_\varepsilon, \delta)$ denote the grid given by Lemma 2.1 with $\delta = \delta(\varepsilon) = |\log \varepsilon|^{-q}$, and defining ν_ε by (5.5) and $\Theta, \partial\mathfrak{G}$ by (2.2), if*

$$(7.1) \quad E_\varepsilon(u_\varepsilon, \Theta) \leq KM_\varepsilon \quad \text{and} \quad \int_{\partial\mathfrak{G}} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2 \leq K\delta^{-1}M_\varepsilon$$

for some universal constant $K > 0$, then

$$E_\varepsilon(u_\varepsilon, \Theta) \geq \frac{1}{2} |\nu_{\varepsilon, \Theta}| \left(\log \frac{1}{\varepsilon} - \log C \frac{M_\varepsilon^{124} |\log \varepsilon|^{(20+1/3)(1+b)}}{\delta^{123}} \right) - \frac{C}{|\log \varepsilon|^b},$$

where C is a constant depending only on $\partial\Omega$.

Proof. We proceed in a similar way to the proof of Proposition 6.1.

Step 1. Lower bound via the co-area formula. For each face $\omega \in \mathfrak{N}_2(\mathfrak{G}(b_\varepsilon, \delta))$ of a cube of the grid such that $\omega \subset \partial\mathfrak{G}$, denote by $\{S_{i,\omega}\}_{i \in I_\omega}$ the collection of connected components of $\{x \in \omega \mid |u_\varepsilon(x)| \leq 1/2\}$. We define

$$S_{\partial\mathfrak{G}} := \cup_{\omega \subset \partial\mathfrak{G}} \cup_{i \in I_\omega} S_{i,\omega}.$$

Note that $|u_\varepsilon(x)| > 1/2$ for any $x \in \partial\mathfrak{G} \setminus S_{\partial\mathfrak{G}}$.

Denote by $\mathcal{A}_{\partial\mathfrak{G}} = \{p_1, \dots, p_{k_{\partial\mathfrak{G}}}, n_1, \dots, n_{k_{\partial\mathfrak{G}}}\} \subset \Omega$ the configuration of positive and negative points associated to $\partial\mathfrak{G}$ defined in (5.6). For parameters $\rho \in (0, 1/4)$ and $0 < \lambda \leq (C_0(2k_{\partial\mathfrak{G}})^{-5})^{1/\rho}$ to be chosen later, let ζ_λ be the smooth function associated to $\mathcal{A}_{\partial\mathfrak{G}}$ by Proposition 5.2, or, more precisely, the function described in Section 5.2.1 for which (up to a relabeling of the points) the quantity $L_{\partial\Omega}(\mathcal{A}_{\partial\mathfrak{G}})$ is replaced with $L_{\partial\Omega}(\tilde{\mathcal{A}}_{\partial\mathfrak{G}})$ in the statement of the proposition. Here the constant $C_0 = C_0(\partial\Omega)$ is the constant appearing in the proposition. In addition, for $0 < \kappa = \kappa(\lambda) < \lambda^{2\rho}/3$ to be chosen later, consider the set T_κ defined there.

Let

$$\partial\tilde{\mathfrak{G}} = \{x \in \Omega \setminus \Theta \mid \min_{y \in \partial\mathfrak{G}} \|x - y\|_\infty = C_1 \lambda^2 \kappa\},$$

where C_1 is the constant appearing in the fourth statement of Proposition 5.2. Observe that $\partial\tilde{\mathfrak{G}}$ corresponds to a shrunk version of the polyhedron $\partial\mathfrak{G}$, or, in other words, a smaller version of $\partial\mathfrak{G}$ with the same shape. Each face $\omega \subset \partial\mathfrak{G}$ has a parallel counterpart face $\tilde{\omega} \subset \partial\tilde{\mathfrak{G}}$ which corresponds to a translated and in some cases also a shrunk version of ω . It is easy to see that there exists a bijective function $f : \partial\mathfrak{G} \rightarrow \partial\tilde{\mathfrak{G}}$ mapping any $x \in \omega \subset \partial\mathfrak{G}$ to its unique counterpart point $\tilde{x} \in \tilde{\omega} \subset \partial\tilde{\mathfrak{G}}$. One immediately checks that, for any $x, y \in \omega \subset \partial\mathfrak{G}$,

$$|f(x) - x| \leq \sqrt{2}C_1 \lambda^2 \kappa \quad \text{and} \quad \frac{1}{\sqrt{2}}|x - y| \leq |f(x) - f(y)| \leq |x - y|.$$

Denoting by \mathcal{O} the open region enclosed by $\partial\mathfrak{G}$ and $\partial\tilde{\mathfrak{G}}$, we observe that for any $y \in \overline{\mathcal{O}}$ there exists a unique $x_y \in \partial\mathfrak{G}$ and a unique $t_y \in [0, 1]$ such that $y = tx + (1 - t)f(x)$. Letting

$$\tilde{\mathcal{O}} := \{y \in \mathcal{O} \mid x_y \notin S_{\partial\mathfrak{G}}\},$$

we define $v_\varepsilon : \overline{\Theta \cup \tilde{\mathcal{O}}} \rightarrow \mathbb{C}$ by

$$v_\varepsilon(y) = u_\varepsilon(y) \quad \text{if } y \in \overline{\Theta}, \quad v_\varepsilon(y) = u_\varepsilon(x_y) \quad \text{if } y \in \overline{\tilde{\mathcal{O}}}.$$

Note that v_ε is a H^1 -extension of u_ε and that

$$E_\varepsilon(v_\varepsilon, \tilde{\mathcal{O}}) \leq E_\varepsilon(v_\varepsilon, \mathcal{O}) \leq \sqrt{2}C_1 \lambda^2 \kappa \int_{\partial\tilde{\mathfrak{G}}} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 \leq 2C_1 \lambda^2 \kappa \int_{\partial\mathfrak{G}} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2.$$

Thus

$$E_\varepsilon(u_\varepsilon, \Theta) \geq E_\varepsilon(v_\varepsilon, \bar{\Theta} \cup \tilde{\mathcal{O}}) - E_\varepsilon(v_\varepsilon, \tilde{\mathcal{O}}) \geq E_\varepsilon(v_\varepsilon, \bar{\Theta} \cup \tilde{\mathcal{O}}) - 2C_1\lambda^2\kappa \int_{\partial\mathfrak{S}} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2.$$

In particular, if $\lambda^2\kappa$ is small enough then $E_\varepsilon(v_\varepsilon, \bar{\Theta} \cup \tilde{\mathcal{O}}) \leq 2M_\varepsilon$. We also define

$$U_\lambda := \zeta_\lambda(\{y \in \partial(\bar{\Theta} \cup \tilde{\mathcal{O}}) \setminus \partial\Omega \mid x_y \in S_{\partial\mathfrak{S}}\})$$

and note that by (4.4), (4.5), and (5.7), we have

$$|U_\lambda| \leq |S_{\partial\mathfrak{S}}| + \sqrt{2}(2\tilde{k}_{\partial\Omega})C_1\lambda^2\kappa \leq C\varepsilon\delta^{-1}M_\varepsilon + C\delta^{-1}M_\varepsilon\lambda^2\kappa.$$

Since $|\nabla\zeta_\lambda| \leq 1$ in Ω_λ , where Ω_λ is the set defined by (5.3), using the co-area formula, we deduce that

$$E_\varepsilon(v_\varepsilon, \bar{\Theta} \cup \tilde{\mathcal{O}}) \geq \int_{(\bar{\Theta} \cup \tilde{\mathcal{O}}) \cap \Omega_\lambda} e_\varepsilon(v_\varepsilon) |\nabla\zeta_\lambda| = \int_{t \in \mathbb{R}} \int_{\{\zeta_\lambda=t\} \cap (\bar{\Theta} \cup \tilde{\mathcal{O}}) \cap \Omega_\lambda} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 dt.$$

We remark that if δ and $\lambda^2\kappa$ are small enough then

$$(\bar{\Theta} \cup \tilde{\mathcal{O}}) \cap \Omega_\lambda = (\bar{\Theta} \cup \tilde{\mathcal{O}}) \cap \tilde{\Omega}_\lambda,$$

where $\tilde{\Omega}_\lambda := \{x \in \Omega \mid 2\lambda^\rho < d(x, \partial\Omega)\}$.

Step 2. Lower bound via the ball construction on a surface. We would now like to apply the results of Section 3. Let us consider a small number $\gamma > 0$ and define

$$V_\gamma := \left\{ t \in \mathbb{R} \mid \int_{\{\zeta_\lambda=t\} \cap (\bar{\Theta} \cup \tilde{\mathcal{O}}) \cap \tilde{\Omega}_\lambda} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 > \frac{1}{\gamma} M_\varepsilon \right\}.$$

Note that $|V_\gamma| \leq 2K\gamma$. Finally, let us define

$$T_{\text{bad}} = T_\kappa \cup U_\lambda \cup V_\gamma \cup \zeta_\lambda(\Omega \setminus \tilde{\Omega}_\lambda),$$

$\Sigma_t := \{\zeta_\lambda = t\} \cap (\bar{\Theta} \cup \tilde{\mathcal{O}}) \cap \tilde{\Omega}_\lambda$, $t_* := \min_{a_i \in \mathcal{A}_{\partial\mathfrak{S}}} \zeta_\lambda(a_i)$, and $t^* := \max_{a_i \in \mathcal{A}_{\partial\mathfrak{S}}} \zeta_\lambda(a_i)$. For $t \in T_{\text{good}} := [t_*, t^*] \setminus T_{\text{bad}}$ it holds that:

- $\int_{\Sigma_t} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 \leq \gamma^{-1} M_\varepsilon$.
- $\{\zeta_\lambda = t\} = \{x \in \tilde{\Omega}_\lambda \mid \zeta_\lambda(x) = t\}$ is a surface whose second fundamental form is bounded by $C_1(\lambda^2\kappa)^{-1}$. Note that this surface is necessarily oriented since it is a level set of ζ_λ .
- $\Sigma_t = \{\zeta_\lambda = t\} \cap (\bar{\Theta} \cup \tilde{\mathcal{O}})$ and $\partial\Sigma_t = \{\zeta_\lambda = t\} \cap (\partial(\bar{\Theta} \cup \tilde{\mathcal{O}}) \setminus \partial\Omega)$.
- $|v_\varepsilon(x)| > 1/2$ if $d(x, \partial\Sigma_t) < C_1\lambda^2\kappa$.

Then Corollary 3.1 applied to v_ε on Σ_t with $Q_\varepsilon = C_1(\lambda^2\kappa)^{-1}$ and $\mathcal{M} = \gamma^{-1}M_\varepsilon$, yields that, for any $t \in T_{\text{good}}$,

$$\int_{\Sigma_t} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 \geq \pi |\deg(v_\varepsilon, \partial\Sigma_t)| \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right).^2$$

²To apply the corollary we actually need $\lambda^{-1}, \kappa^{-1}$, and γ^{-1} to be bounded above by positive powers of $|\log \varepsilon|$. For this reason, and because of our choice of these parameters in terms of $\delta(\varepsilon)$ (see Step 3), we require δ to be a negative power of $|\log \varepsilon|$ in the statement.

We point out that we cannot directly apply Corollary 3.1 to u_ε on $\Sigma = \{\zeta_\lambda = t\} \cap \Theta$ (for $t \in T_{\text{good}}$), because (3.5) does not necessarily hold in this case. For this reason, we extended the function u_ε to $\overline{\Theta \cup \tilde{\mathcal{O}}}$ in the previous step.

Noting that $\partial\Sigma_t = \partial(\{\zeta_\lambda \geq t\} \cap (\partial(\overline{\Theta \cup \tilde{\mathcal{O}}}) \setminus \partial\Omega))$, we deduce that

$$\deg(v_\varepsilon, \partial\Sigma_t) = d(t) := \#\{i \mid \zeta_\lambda(p_i) > t\} - \#\{i \mid \zeta_\lambda(n_i) > t\}.$$

By combining our previous estimates, we find

$$\begin{aligned} E_\varepsilon(v_\varepsilon, \overline{\Theta \cup \tilde{\mathcal{O}}}) &\geq \int_{t \in T_{\text{good}}} \int_{\Sigma_t} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 dt \\ &\geq \pi \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right) \int_{t \in T_{\text{good}}} d(t) dt \\ &\geq \pi \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right) \left(\int_{t_*}^{t^*} d(t) dt - \int_{t \in T_{\text{bad}}} |d(t)| dt \right). \end{aligned}$$

Moreover, for any $t \in T_{\text{bad}}$,

$$|d(t)| = |\#\{i \mid \zeta_\lambda(p_i) > t\} - \#\{i \mid \zeta_\lambda(n_i) > t\}| \leq \tilde{k}_{\partial\mathfrak{G}}.$$

Then

$$\int_{t \in T_{\text{bad}}} |d(t)| dt \leq \tilde{k}_{\partial\mathfrak{G}} |T_{\text{bad}}| \leq \tilde{k}_{\partial\mathfrak{G}} (|T_\kappa| + |U_\lambda| + |V_\gamma| + |\zeta_\lambda(\{x \in \Omega \mid d(x, \partial\Omega) \leq 2\lambda^\rho\})|).$$

On the other hand, observe that

$$\int_{t_*}^{t^*} d(t) dt = \int_{t_*}^{t^*} (\#\{i \mid \zeta_\lambda(p_i) > t\} - \#\{i \mid \zeta_\lambda(n_i) > t\}) dt = \sum_{i=1}^{\tilde{k}_{\partial\mathfrak{G}}} \zeta_\lambda(p_i) - \zeta_\lambda(n_i).$$

Since

$$\left| L_{\partial\Omega}(\tilde{\mathcal{A}}_{\partial\mathfrak{G}}) - \sum_{i=1}^{\tilde{k}_{\partial\mathfrak{G}}} \zeta_\lambda(p_i) - \zeta_\lambda(n_i) \right| \leq C(2\tilde{k}_{\partial\mathfrak{G}})^6 \lambda^\rho,$$

and remembering that $|2\pi L_{\partial\Omega}(\tilde{\mathcal{A}}_{\partial\mathfrak{G}}) - |\nu_{\varepsilon, \Theta}|$ can be taken arbitrarily small, we conclude that

$$\int_{t_*}^{t^*} d(t) dt \geq \frac{1}{2\pi} |\nu_{\varepsilon, \Theta}| - C(2\tilde{k}_{\partial\mathfrak{G}})^6 \lambda^\rho.$$

Collecting our previous computations, we find

$$E_\varepsilon(u_\varepsilon, \Theta) \geq \frac{1}{2} |\nu_{\varepsilon, \Theta}| \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right) - \mathcal{E}_\Theta,$$

where

$$\begin{aligned} \mathcal{E}_\Theta := & C \left((2\tilde{k}_{\partial\mathfrak{G}})^6 \lambda^\rho + \tilde{k}_{\partial\mathfrak{G}} (|T_\kappa| + |U_\lambda| + |V_\gamma| + |\zeta_\lambda(\{x \in \Omega \mid d(x, \partial\Omega) \leq 2\lambda^\rho\})|) \right) \log \frac{1}{\varepsilon} \\ & + 2C_1 \lambda^2 \kappa \int_{\partial\mathfrak{G}} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2. \end{aligned}$$

Step 3. Choice of the parameters. We now choose the parameters ρ , λ , $\kappa(\lambda)$ and γ . Observe that (4.4) and (5.7) imply that

$$(2\tilde{k}_{\partial\mathfrak{G}}) \leq C\delta^{-1}M_\varepsilon.$$

Letting $\kappa = \lambda^{2\rho}/6$ and using (7.1), we are led to

$$\mathcal{E}_\Theta \leq C \log \frac{1}{\varepsilon} \left(\frac{M_\varepsilon^6}{\delta^6} \lambda^\rho + \frac{M_\varepsilon^4}{\delta^4} \lambda^{3\rho/4} + \frac{M_\varepsilon^9}{\delta^9} \lambda^{1-2\rho} + \frac{M_\varepsilon^7}{\delta^7} \lambda^{1-8\rho} + \frac{M_\varepsilon^2}{\delta^2} \varepsilon + \frac{M_\varepsilon}{\delta} \gamma \right).$$

Then, choosing $\rho = 6/55$,

$$\lambda = \left(\frac{1}{|\log \varepsilon|^{1+b}} \frac{\delta^6}{M_\varepsilon^6} \right)^{\frac{55}{6}}, \quad \text{and} \quad \gamma = \frac{1}{|\log \varepsilon|^{1+b}} \frac{\delta}{M_\varepsilon},$$

we easily check that there exists $\varepsilon_0 > 0$ depending only on b, q, m, M , and $\partial\Omega$, such that $\mathcal{E}_\Theta \leq C|\log \varepsilon|^{-b}$ for any $0 < \varepsilon < \varepsilon_0$. Thus

$$E_\varepsilon(u_\varepsilon, \Theta) \geq \frac{1}{2} |\nu_{\varepsilon, \Theta}| \left(\log \frac{1}{\varepsilon} - \log C \frac{M_\varepsilon^{124} |\log \varepsilon|^{(20+1/3)(1+b)}}{\delta^{123}} \right) - \frac{C}{|\log \varepsilon|^b}.$$

This concludes the proof of the proposition. \square

Remark 7.1. We remark that if in addition $\partial\Omega$ has strictly positive Gauss curvature then we can use the smooth approximation of the function ζ for $d_{\partial\Omega}$ constructed in Appendix C. By using Proposition 5.3 with $\tau = \delta$ and arguing similarly as above, one can prove that

$$E_\varepsilon(u_\varepsilon, \Theta) \geq \frac{1}{2} |\nu_{\varepsilon, \Theta}| \left(\log \frac{1}{\varepsilon} - \log C \frac{M_\varepsilon^{56} |\log \varepsilon|^{7(1+b)}}{\delta^{55}} \right) - C\delta M_\varepsilon |\log \varepsilon| - \frac{C}{|\log \varepsilon|^b}.$$

8. PROOF OF THE MAIN RESULTS

8.1. Proof of Theorem 1.1. First, using the results of the previous two sections we prove (1.1).

Proof of (1.1). Since the energy $F_\varepsilon(u_\varepsilon, A_\varepsilon)$ is gauge invariant, it is enough to prove the result in the Coulomb gauge, i.e.

$$\operatorname{div} A_\varepsilon = 0 \text{ in } \Omega \quad \text{and} \quad A_\varepsilon \cdot \nu = 0 \text{ on } \partial\Omega.$$

We immediately check that

$$\|A_\varepsilon\|_{H^1(\Omega, \mathbb{R}^3)} \leq C \|\operatorname{curl} A_\varepsilon\|_{L^2(\Omega, \mathbb{R}^3)},$$

where throughout the proof $C > 0$ denotes a universal constant that may change from line to line. By Sobolev embedding theorem we have

$$\|A_\varepsilon\|_{L^p(\Omega, \mathbb{R}^3)} \leq C \|A_\varepsilon\|_{H^1(\Omega, \mathbb{R}^3)}$$

for any $1 \leq p \leq 6$. Observe that

$$\int_{\Omega} |\nabla u_\varepsilon|^2 \leq \int_{\Omega} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + |u_\varepsilon|^2 |A_\varepsilon|^2 \leq \int_{\Omega} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + (|u_\varepsilon|^2 - 1) |A_\varepsilon|^2 + |A_\varepsilon|^2.$$

By the Cauchy-Schwarz inequality, we have

$$\int_{\Omega} (|u_\varepsilon|^2 - 1) |A_\varepsilon|^2 \leq \left(\int_{\Omega} (1 - |u_\varepsilon|^2)^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |A_\varepsilon|^4 \right)^{\frac{1}{2}} \leq C \varepsilon F_\varepsilon(u_\varepsilon, A_\varepsilon).$$

Thus

$$E_\varepsilon(u_\varepsilon) \leq C F_\varepsilon(u_\varepsilon, A_\varepsilon).$$

Let us consider the grid $\mathfrak{G}(b_\varepsilon, \delta)$ given by Lemma 2.1. It is not hard to see that, up to an adjustment of the constant appearing in the lemma, we can require our grid to additionally satisfy the inequalities

$$(8.1) \quad \int_{\mathfrak{R}_1(\mathfrak{G}(b_\varepsilon, \delta))} e_\varepsilon(u_\varepsilon) d\mathcal{H}^1 \leq C \delta^{-2} F_\varepsilon(u_\varepsilon, A_\varepsilon), \quad \int_{\mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, \delta))} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2 \leq C \delta^{-1} F_\varepsilon(u_\varepsilon, A_\varepsilon).$$

We define the polyhedral 1-current ν_ε by (5.5). We recall the notation introduced in Lemma 5.5 and observe that

$$\int_{S_{\nu_\varepsilon}} |\nabla u_\varepsilon|^2 \leq \int_{S_{\nu_\varepsilon}} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \int_{S_{\nu_\varepsilon}} (|u_\varepsilon|^2 - 1) |A_\varepsilon|^2 + \int_{S_{\nu_\varepsilon}} |A_\varepsilon|^2.$$

Using Hölder's inequality, we find

$$\int_{S_{\nu_\varepsilon}} (|u_\varepsilon|^2 - 1) |A_\varepsilon|^2 \leq \| |u_\varepsilon|^2 - 1 \|_{L^2(S_{\nu_\varepsilon})} |S_{\nu_\varepsilon}|^{\frac{1}{6}} \|A_\varepsilon\|_{L^6(S_{\nu_\varepsilon}, \mathbb{R}^3)}^2$$

and

$$\int_{S_{\nu_\varepsilon}} |A_\varepsilon|^2 \leq |S_{\nu_\varepsilon}|^{\frac{2}{3}} \|A_\varepsilon\|_{L^6(S_{\nu_\varepsilon}, \mathbb{R}^3)}^2.$$

We are led to

$$\int_{S_{\nu_\varepsilon}} |\nabla u_\varepsilon|^2 \leq \int_{S_{\nu_\varepsilon}} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + C F_\varepsilon(u_\varepsilon, A_\varepsilon) \left(\varepsilon |S_{\nu_\varepsilon}|^{\frac{1}{6}} + |S_{\nu_\varepsilon}|^{\frac{2}{3}} \right),$$

which implies that

$$(8.2) \quad \frac{1}{2} \int_{S_{\nu_\varepsilon}} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 \geq E_\varepsilon(u_\varepsilon, S_{\nu_\varepsilon}) - C F_\varepsilon(u_\varepsilon, A_\varepsilon) \left(\varepsilon |S_{\nu_\varepsilon}|^{\frac{1}{6}} F_\varepsilon(u_\varepsilon, A_\varepsilon)^{\frac{1}{2}} + |S_{\nu_\varepsilon}|^{\frac{2}{3}} \right).$$

Thanks to (8.1), we can apply Proposition 6.1 and Proposition 7.1 with $b = n$ and $q > 0$ (in particular $\delta = \delta(\varepsilon) = |\log \varepsilon|^{-q}$). We then deduce that there exists $\varepsilon_0 > 0$, depending only on q, m, n, M , and $\partial\Omega$, such that, for any $0 < \varepsilon < \varepsilon_0$,

$$E_\varepsilon(u_\varepsilon, S_{\nu_\varepsilon}) \geq \frac{1}{2} |\nu_\varepsilon|(\Omega) \left(\log \frac{1}{\varepsilon} - \log C \frac{M_\varepsilon^{124} |\log \varepsilon|^{(20+1/3)(1+n)}}{\delta^{123}} \right) - \frac{C}{|\log \varepsilon|^n},$$

where C is a constant depending only on $\partial\Omega$. By combining this with (8.2) and Lemma 5.5, we are led to

$$\begin{aligned} \int_{S_{\nu_\varepsilon}} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 + |\operatorname{curl} A_\varepsilon|^2 \\ \geq |\nu_\varepsilon|(\Omega) \left(\log \frac{1}{\varepsilon} - \log C \frac{M_\varepsilon^{124} |\log \varepsilon|^{(20+1/3)(1+n)}}{\delta^{123}} \right) \\ - CM_\varepsilon \delta^{\frac{2}{3}} - \frac{C}{|\log \varepsilon|^n}. \end{aligned}$$

By letting $q = q(m, n) = \frac{3}{2}(m + n)$, we have $M_\varepsilon \delta^{\frac{2}{3}} \leq C |\log \varepsilon|^{-n}$. This concludes the proof of the lower bound. \square

Before presenting the proof of (1.2) for $\gamma = 1$, let us prove the following lemma.

Lemma 8.1. *Let $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$. Then there exists a constant $C > 0$ depending only on $\partial\Omega$, such that*

$$(8.3) \quad \|\mu(u_\varepsilon, A_\varepsilon)\|_{C^0(\Omega)^*} \leq CF_\varepsilon(u_\varepsilon, A_\varepsilon).$$

Proof. By definition

$$\begin{aligned} \mu(u_\varepsilon, A_\varepsilon) &= \frac{i}{2} d(u_\varepsilon d_{A_\varepsilon} \bar{u}_\varepsilon - \bar{u}_\varepsilon d_{A_\varepsilon} u_\varepsilon) + dA_\varepsilon \\ &= \frac{i}{2} (du_\varepsilon \wedge d_{A_\varepsilon} \bar{u}_\varepsilon + u_\varepsilon d(d_{A_\varepsilon} \bar{u}_\varepsilon) - d\bar{u}_\varepsilon \wedge d_{A_\varepsilon} u_\varepsilon - \bar{u}_\varepsilon d(d_{A_\varepsilon} u_\varepsilon)) + dA_\varepsilon. \end{aligned}$$

Simple computations show that

$$\begin{aligned} \mu(u_\varepsilon, A_\varepsilon) &= \frac{i}{2} (du_\varepsilon \wedge d_{A_\varepsilon} \bar{u}_\varepsilon - iu_\varepsilon d\bar{u}_\varepsilon \wedge A_\varepsilon - d\bar{u}_\varepsilon \wedge d_{A_\varepsilon} u_\varepsilon + i\bar{u}_\varepsilon du_\varepsilon \wedge A_\varepsilon) + dA_\varepsilon \\ &= \frac{i}{2} (d_{A_\varepsilon} u_\varepsilon \wedge d_{A_\varepsilon} \bar{u}_\varepsilon - d_{A_\varepsilon} \bar{u}_\varepsilon \wedge d_{A_\varepsilon} u_\varepsilon) + dA_\varepsilon = id_{A_\varepsilon} u_\varepsilon \wedge d_{A_\varepsilon} \bar{u}_\varepsilon + dA_\varepsilon. \end{aligned}$$

Integrating on Ω and using the Cauchy-Schwarz inequality, we find

$$\int_{\Omega} \mu(u_\varepsilon, A_\varepsilon) \leq 2 \left(F_\varepsilon(u_\varepsilon, A_\varepsilon) + F_\varepsilon(u_\varepsilon, A_\varepsilon)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \right).$$

Then we easily check that there exists a constant $C(\partial\Omega) > 0$ such that

$$\left| \int_{\Omega} \mu(u_\varepsilon, A_\varepsilon) \wedge \phi \right| \leq C \|\phi\|_{C^0(\Omega)} F_\varepsilon(u_\varepsilon, A_\varepsilon)$$

for any continuous 1-form ϕ , which implies (8.3). \square

Proof of (1.2) for $\gamma = 1$. As in the proof of (1.1), we consider the grid $\mathfrak{G}(b_\varepsilon, \delta)$ given by Lemma 2.1 and the polyhedral 1-current ν_ε defined by (5.5). The parameter δ is defined as above.

Let $\phi \in C_T^{0,1}(\Omega)$ be a 1-form. Note that

$$(8.4) \quad \left| \int_{\Omega} (\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon) \wedge \phi \right| \leq \sum_{\mathcal{C}_l \in \mathfrak{G}(b_\varepsilon, \delta)} \left| \int_{\mathcal{C}_l} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}) \wedge \phi \right| + \left| \int_{\Theta} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}) \wedge \phi \right|.$$

First, we consider a cube $\mathcal{C}_l \in \mathfrak{G}(b_\varepsilon, \delta)$ and define $\phi_l = \int_{\mathcal{C}_l} \phi$. Observe that

$$(8.5) \quad \|\phi - \phi_l\|_{C^0(\mathcal{C}_l)} \leq \delta \|\phi\|_{C^{0,1}(\mathcal{C}_l)}$$

and that

$$\left| \int_{\mathcal{C}_l} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}) \wedge \phi \right| \leq \left| \int_{\mathcal{C}_l} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}) \wedge (\phi - \phi_l) \right| + \left| \int_{\mathcal{C}_l} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}) \wedge \phi_l \right|.$$

Using (8.5), we deduce that

$$(8.6) \quad \left| \int_{\mathcal{C}_l} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}) \wedge (\phi - \phi_l) \right| \leq \delta \|\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}\|_{C^0(\mathcal{C}_l)^*} \|\phi\|_{C^{0,1}(\mathcal{C}_l)}.$$

On the other hand, since ϕ_l is a constant, there exist a function f_l such that

$$\phi_l = df_l, \quad \int_{\mathcal{C}_l} f_l = 0.$$

In particular

$$\|f_l\|_{C^{0,1}(\mathcal{C}_l)} \leq |\phi_l|.$$

By an integration by parts, we have

$$\int_{\mathcal{C}_l} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}) \wedge \phi_l = \sum_{\omega \subset \partial \mathcal{C}_l} \int_{\omega} \left(\mu_{\varepsilon, \omega} - 2\pi \sum_{i \in I_\omega} d_{i, \omega} \delta_{a_{i, \omega}} \right) f_l.$$

Here, we have used the notation introduced in Section 4 and the fact that the restriction of $\nu_{\varepsilon, \mathcal{C}_l}$ to each of the six faces ω of the cube \mathcal{C}_l is equal to $2\pi \sum_{i \in I_\omega} d_{i, \omega} \delta_{a_{i, \omega}}$. Corollary 4.1 then yields that

$$(8.7) \quad \left| \int_{\mathcal{C}_l} (\mu_\varepsilon(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}) \wedge \phi_l \right| \leq C_0 \sum_{\omega \subset \partial \mathcal{C}_l} \max(r_\omega, \varepsilon) \left(1 + \int_{\omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^2 + \int_{\partial \omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^1 \right) \|f_l\|_{C^{0,1}(\mathcal{C}_l)},$$

where throughout the proof C_0 denotes a universal constant that may change from line to line. Using (8.6) and (8.7), we deduce that

$$\begin{aligned} \|\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{G}_l}\|_{C^{0,1}(\mathcal{G}_l)^*} &\leq \delta \|\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{G}_l}\|_{C^0(\mathcal{G}_l)^*} \\ &\quad + C_0 \sum_{\omega \subset \partial \mathcal{G}_l} \max(r_\omega, \varepsilon) \left(1 + \int_\omega e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^2 + \int_{\partial \omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^1 \right) \end{aligned}$$

for any cube $\mathcal{G}_l \in \mathfrak{G}(b_\varepsilon, \delta)$. Then by summing over the cubes of the grid, we obtain

$$\begin{aligned} \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^{0,1}(\Omega \setminus \Theta)^*} &\leq \delta \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^0(\Omega \setminus \Theta)^*} \\ &\quad + C_0 \max(r_\mathfrak{G}, \varepsilon) \left(1 + 2 \int_{\mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, \delta))} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^2 + 8 \int_{\mathfrak{R}_1(\mathfrak{G}(b_\varepsilon, \delta))} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^1 \right). \end{aligned}$$

Using (2.1b), (2.1c), and (4.5), we find

$$(8.8) \quad \begin{aligned} \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^{0,1}(\Omega \setminus \Theta)^*} &\leq \delta \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^0(\Omega \setminus \Theta)^*} \\ &\quad + C_0 \max(\varepsilon \delta^{-1} F_\varepsilon(u_\varepsilon, A_\varepsilon), \varepsilon) (1 + \delta^{-2} F_\varepsilon(u_\varepsilon, A_\varepsilon)). \end{aligned}$$

We now provide an estimate for the last term in (8.4). Observe that if ε is sufficiently small, and since $\partial\Omega$ is of class C^2 , for any $y \in \overline{\Theta}$, there exists a unique $x_y = \text{proj}_{\partial\Omega} y$ such that $y = x_y - t_y \nu(x_y)$, for some $t_y \geq 0$, where $\nu(x_y)$ is the outer unit normal to $\partial\Omega$ at x_y . We define $f : \Theta \rightarrow \mathbb{R}$ by

$$f(y) = f(x_y - t_y \nu(x_y)) = -t_y \phi(x_y) \cdot \nu(x_y).$$

By noting that, for any $y \in \Theta$,

$$\nabla f(y) = (\phi(x_y) \cdot \nu(x_y)) \nu(x_y) = \phi(x_y),$$

one can easily check that

$$\|f\|_{C^{0,1}(\Theta)} \leq \|\phi\|_{C^{0,1}(\Theta)} \quad \text{and} \quad \|\phi - \nabla f\|_{C^0(\Theta)} \leq \delta \|\phi\|_{C^{0,1}(\Theta)}.$$

We now write

$$\int_\Theta (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}) \wedge \phi = \int_\Theta (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}) \wedge (\phi - df) + \int_\Theta (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}) \wedge df.$$

Observe that

$$\left| \int_\Theta (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}) \wedge (\phi - df) \right| \leq \|\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}\|_{C^0(\Theta)^*} \|\phi - df\|_{C^0(\Theta)}.$$

On the other hand, by an integration by parts, we find

$$\int_\Theta (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}) \wedge df = \sum_{\omega \subset \partial \mathfrak{G}} \int_\omega (\mu_{\varepsilon, \omega} - 2\pi \sum_{i \in I_\omega} d_{i, \omega} \delta_{a_{i, \omega}}) f.$$

Here, we have used the fact that the restriction of $\nu_{\varepsilon, \Theta}$ to each of the faces ω of a cube of the grid such that $\omega \subset \partial \mathfrak{G}$ is equal to $2\pi \sum_{i \in I_\omega} d_{i, \omega} \delta_{a_{i, \omega}}$. We then deduce that

$$\begin{aligned} \left| \int_{\Theta} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}) \wedge \phi \right| &\leq \delta \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^0(\Theta)^*} \\ &\quad + C_0 \max(\varepsilon \delta^{-1} F_\varepsilon(u_\varepsilon, A_\varepsilon), \varepsilon) (1 + \delta^{-2} F_\varepsilon(u_\varepsilon, A_\varepsilon)) \|\phi\|_{C^{0,1}(\Theta)}. \end{aligned}$$

By combining this with (8.4) and (8.8), we find

$$\begin{aligned} \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C_T^{0,1}(\Omega)^*} &\leq \delta \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^0(\Omega)^*} \\ &\quad + C_0 \max(\varepsilon \delta^{-1} F_\varepsilon(u_\varepsilon, A_\varepsilon), \varepsilon) (1 + \delta^{-2} F_\varepsilon(u_\varepsilon, A_\varepsilon)). \end{aligned}$$

Observe now that

$$\|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^0(\Omega)^*} \leq \|\mu(u_\varepsilon, A_\varepsilon)\|_{C^0(\Omega)^*} + \|\nu_\varepsilon\|_{C^0(\Omega)^*}.$$

From (1.1), we deduce that

$$\|\nu_\varepsilon\|_{C^0(\Omega)^*} \leq C_0 \frac{F_\varepsilon(u_\varepsilon, A_\varepsilon)}{|\log \varepsilon|}$$

By combining the previous two estimates with (8.3), we get

$$(8.9) \quad \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^0(\Omega)^*} \leq C F_\varepsilon(u_\varepsilon, A_\varepsilon),$$

where C is a constant depending only on $\partial \Omega$. This implies that

$$(8.10) \quad \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C_T^{0,1}(\Omega)^*} \leq C \delta F_\varepsilon(u_\varepsilon, A_\varepsilon) + C_0 \max(\varepsilon \delta^{-1} F_\varepsilon(u_\varepsilon, A_\varepsilon), \varepsilon) (1 + \delta^{-2} F_\varepsilon(u_\varepsilon, A_\varepsilon)).$$

From this, (1.2) for $\gamma = 1$ follows. \square

The proof of (1.2) for $\gamma \in (0, 1)$ uses the following simple interpolation fact, as in [JS02].

Lemma 8.2. *Assume μ is a Radon measure on Ω . Then for any $\gamma \in (0, 1)$,*

$$\|\mu\|_{C_0^{0,\gamma}(\Omega)^*} \leq \|\mu\|_{C_0^0(\Omega)^*}^{1-\gamma} \|\mu\|_{C_0^{0,1}(\Omega)^*}^\gamma.$$

Proof of (1.2) for $\gamma \in (0, 1)$. Note that $\|\mu\|_{C_0^{0,\gamma}(\Omega)^*} \leq \|\mu\|_{C_T^{0,\gamma}(\Omega)^*}$ for any 1-current μ . By combining the previous lemma with (8.9) and (8.10), we are led to

$$\|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C_0^{0,\gamma}(\Omega)^*} \leq C F_\varepsilon(u_\varepsilon, A_\varepsilon)^{1-\gamma} \delta^\gamma (F_\varepsilon(u_\varepsilon, A_\varepsilon) + 1)^\gamma \leq C \delta^\gamma (F_\varepsilon(u_\varepsilon, A_\varepsilon) + 1)$$

for any $\gamma \in (0, 1)$, where $C > 0$ is a constant depending only on γ and $\partial \Omega$. Then the proof reduces to proving that this estimate is still valid when we replace the norm $\|\cdot\|_{C_0^{0,\gamma}(\Omega)^*}$ with $\|\cdot\|_{C_T^{0,\gamma}(\Omega)^*}$. Arguing as in the proof of [JMS04, Proposition 3.1], we conclude that (1.2) holds for $\gamma \in (0, 1)$. \square

8.2. Proof of Theorem 1.2.

Proof. As in the proof of Theorem 1.1, we consider the grid $\mathfrak{G}(b_\varepsilon, \delta)$ given by Lemma 2.1 and the polyhedral 1-current ν_ε defined by (5.5).

Let us first prove the lower bound. The main difference with the proof of (1.1) is that in this case we cannot use Proposition 7.1 and therefore we cannot provide a lower bound for the free energy close to the boundary. Moreover, by arguing in the same fashion as before, we immediately check that

$$(8.11) \quad \int_{S_{\nu_\varepsilon}} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 + |\operatorname{curl} A_\varepsilon|^2 \geq |\nu_\varepsilon|(\Omega \setminus \Theta) \left(\log \frac{1}{\varepsilon} - \log C \frac{M_\varepsilon^{56} |\log \varepsilon|^{7(1+n)}}{\delta^{55}} \right) - CM_\varepsilon \delta^{\frac{2}{3}} - C |\log \varepsilon|^{-n},$$

where $\delta = \delta(\varepsilon) = |\log \varepsilon|^{-q}$ with $q > 0$. Choosing once again $q = \frac{3}{2}(m+n)$, and noting that by the definition of Θ (recall (2.2)) we have $\Omega_\varepsilon \subset \Omega \setminus \Theta$, we get the lower bound.

We now prove the vorticity estimate for $\gamma = 1$. In this case, we work in the space $C_0^{0,1}(\Omega)^*$ instead of $C_T^{0,1}(\Omega)^*$. Let $\phi \in C_0^{0,1}(\Omega)$ be a 1-form. We begin by observing that (8.8) also holds in this case.

Since $\phi = 0$ on $\partial\Omega$, we have that

$$\|\phi\|_{C^0(\Theta)} \leq C_0 \delta \|\phi\|_{C^{0,1}(\Theta)},$$

and therefore

$$(8.12) \quad \left| \int_{\Theta} (\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon) \wedge \phi \right| \leq C_0 \delta \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^0(\Theta)^*} \|\phi\|_{C^{0,1}(\Theta)}.$$

On the other hand, from (8.11), we have

$$\|\nu_\varepsilon\|_{C^0(\Omega \setminus \Theta)^*} \leq C_0 \frac{F_\varepsilon(u_\varepsilon, A_\varepsilon)}{|\log \varepsilon|}.$$

From (4.4), we deduce that

$$\|\nu_\varepsilon\|_{C^0(\Theta)^*} \leq C_0 \delta \frac{F_\varepsilon(u_\varepsilon, A_\varepsilon)}{\delta} = C_0 F_\varepsilon(u_\varepsilon, A_\varepsilon).$$

Therefore

$$\|\nu_\varepsilon\|_{C^0(\Omega)^*} \leq C_0 F_\varepsilon(u_\varepsilon, A_\varepsilon).$$

By combining this with (8.3), we get

$$\|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^0(\Omega)^*} \leq C F_\varepsilon(u_\varepsilon, A_\varepsilon).$$

From this, (8.8), and (8.12) we obtain the vorticity estimate for $\gamma = 1$. The estimate for $\gamma \in (0, 1)$ directly follows from Lemma 8.2. \square

9. PROOF OF THE QUANTITATIVE PRODUCT ESTIMATE

In this section, we use ideas from [Ser17, Appendix A]. As in Section 1.4, we view things in three dimensions, where the first dimension is time and the last two are spatial dimensions.

We consider $X \in C_0^{0,1}([0, T] \times \bar{\omega}, \mathbb{R}^2)$, a compactly supported spatial vector field depending on time, and a function $f \in C_0^{0,1}([0, T] \times \bar{\omega})$. Let K denote the union of the supports of X and f . In order to reduce ourselves to the situation where X and f are locally constants, we use a partition of unity at a small scale: let M_ε be as in (1.3) and let us consider a covering of K by $m(\varepsilon)$ balls of radius $M_\varepsilon^{-1/4}$ (with bounded overlap), and let $\{D_k\}_{k=1}^{m(\varepsilon)}$ be an indexation of this sequence of balls and $\{\chi_k\}_{k=1}^{m(\varepsilon)}$ a partition of unity associated to this covering such that $\sum_{k=1}^{m(\varepsilon)} \chi_k = 1$ and $\|\nabla \chi_k\|_{L^\infty} \leq M_\varepsilon^{1/4}$ for any $k = 1, \dots, m(\varepsilon)$. For each $k \in \{1, \dots, m(\varepsilon)\}$, let then X_k and f_k be the averages of X and f in D_k . Then, working only in D_k , without loss of generality, we can assume that X_k is aligned with the first space coordinate vector e_1 , with (e_t, e_1, e_2) forming an orthonormal frame and the coordinates in that frame being denoted by (t, w, σ) . We will assume first that $X_k, f_k \neq 0$. Let us define for each k, σ the set

$$\Theta_{k,\sigma} = \{(t, w) \mid (t, w, \sigma) \in D_k\},$$

which is a slice of D_k (hence a two dimensional ball). Let us write $\mu_{\varepsilon,k,\sigma}$ for $\mu_\varepsilon(e_t, e_1)$ restricted to $\Theta_{k,\sigma}$. In other words, if ξ is a smooth test-function on $\Omega_{\sigma,k}$, we have

$$(9.1) \quad \int_{\Theta_{k,\sigma}} \mu_{\varepsilon,k,\sigma} \wedge \xi = - \int_{\Theta_{k,\sigma}} (\langle du_\varepsilon - iu_\varepsilon A_\varepsilon, iu_\varepsilon \rangle + A_\varepsilon) \wedge d\xi,$$

where d denotes the differential in the slice $\Theta_{k,\sigma}$.

For a given $\Lambda > 0$, we let g_k be the constant metric on $\Theta_{k,\sigma}$ defined by $g_k(e_t, e_t) = \sqrt{\Lambda}/|f_k|$, $g_k(e_1, e_1) = 1/(\sqrt{\Lambda}|X_k|)$, and $g_k(e_t, e_1) = 0$.

We then apply the ball construction method in each set $\Theta_{k,\sigma}$. Instead of constructing balls for the flat metric, we construct geodesic balls for the constant metric g_k , i.e. here, ellipses.

Lemma 9.1. *Let $\Theta_{k,\sigma} \subset \mathbb{R}^2$ be as above and denote*

$$\Theta_{k,\sigma}^\varepsilon = \{x \in \Theta_{k,\sigma} \mid \text{dist}_{g_k}(x, \partial\Theta_{k,\sigma}) > \varepsilon\}.$$

Assume that

$$F_{\varepsilon,k,\sigma} := \frac{1}{2} \int_{\Theta_{k,\sigma}} |\partial_t u_\varepsilon - iu_\varepsilon \Phi_\varepsilon|^2 + |\partial_w u_\varepsilon - iu_\varepsilon B_{\varepsilon,w}|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 + |\partial_t B_{\varepsilon,w} - \partial_w \Phi_\varepsilon|^2 \leq M_\varepsilon$$

with M_ε as in (1.3). Then if ε is small enough, there exist a universal constant $C > 0$ and a finite collection of disjoint closed balls $B = \{B_i\}_{i \in I}$ for the metric g_k of

centers a_i and radii r_i such that, letting $C_{k,\Lambda} := \max \left\{ \frac{|f_k|}{\Lambda|X_k|}, \frac{\Lambda|X_k|}{|f_k|}, |f_k||X_k|, \frac{1}{|f_k||X_k|} \right\}$

and $R_{k,\Lambda} := \max \left\{ \frac{\sqrt{\Lambda}}{|f_k|} \frac{1}{\sqrt{\Lambda|X_k|}} \right\}$, we have

$$(1) \quad r(B) = \sum_i r_i \leq R_{k,\Lambda} |f_k|^2 |X_k|^2 M_\varepsilon^{-1},$$

$$(2) \quad \text{letting } V = \Theta_{k,\sigma}^\varepsilon \cap \cup_{i \in I} B_i,$$

$$\{|u_\varepsilon(x)| - 1| \geq 1/2\} \cap \Theta_{k,\sigma}^\varepsilon \subset V,$$

(3) writing $d_i = \deg(u_\varepsilon/|u_\varepsilon|, \partial B_i)$ if $B_i \subset \Theta_{k,\sigma}^\varepsilon$ and $d_i = 0$ otherwise, we have for each i ,

$$(9.2) \quad \begin{aligned} \frac{1}{2} \int_{B_i \cap \Theta_{k,\sigma}^\varepsilon} \frac{|f_k|^2}{\Lambda} |\partial_t u_\varepsilon - i u_\varepsilon \Phi_\varepsilon|^2 + \Lambda |X_k|^2 |\partial_w u_\varepsilon - i u_\varepsilon B_{\varepsilon,w}|^2 \\ + |f_k|^5 |X_k|^5 M_\varepsilon^{-2} |\partial_t B_{\varepsilon,w} - \partial_w \Phi_\varepsilon|^2 \\ \geq \pi |d_i| (|\log \varepsilon| - C \log M_\varepsilon + \log(C_{k,\Lambda}^{-1} |f_k|^2 |X_k|^2)) |f_k||X_k|, \end{aligned}$$

(4) and letting $\mu_{\varepsilon,k,\sigma} = 2\pi \sum_i d_i \delta_{a_i}$, we have for any $\xi \in C_0^{0,1}(\Theta_{k,\sigma})$,

$$\left| \int_{\Theta_{k,\sigma}} (J_{\varepsilon,k,\sigma} - \mu_{\varepsilon,k,\sigma}) \wedge \xi \right| \leq C \|\xi\|_{C^{0,1}} C_{k,\Lambda} |f_k|^2 |X_k|^2 M_\varepsilon^{-1} F_{\varepsilon,k,\sigma}.$$

Proof. Let us consider the sets

$$\tilde{\Theta}_{k,\sigma} = \left\{ (s, z) \mid \left(\frac{|f_k|}{\sqrt{\Lambda}} s, \sqrt{\Lambda} |X_k| z \right) \in \Theta_{k,\sigma} \right\}, \quad \tilde{\Theta}_{k,\sigma}^\varepsilon = \{x \in \tilde{\Theta}_{k,\sigma} \mid d(x, \partial \tilde{\Theta}_{k,\sigma}) > \varepsilon\},$$

and define, for $(s, z) \in \tilde{\Theta}_{k,\sigma}$, the function

$$\tilde{u}_\varepsilon(s, z) = u_\varepsilon \left(\frac{|f_k|}{\sqrt{\Lambda}} s, \sqrt{\Lambda} |X_k| z, \sigma \right)$$

and the vector field

$$\tilde{A}_\varepsilon(s, z) = [\tilde{A}_{\varepsilon,s}, \tilde{A}_{\varepsilon,z}](s, z) = \left[\sqrt{\Lambda} |X_k| \Phi_\varepsilon, \frac{|f_k|}{\sqrt{\Lambda}} B_{\varepsilon,w} \right] \left(\frac{|f_k|}{\sqrt{\Lambda}} s, \sqrt{\Lambda} |X_k| z, \sigma \right).$$

The first three items are a consequence of [SS07, Proposition 4.3]. We start by noting that by making the change of variables $t = \frac{\sqrt{\Lambda}}{|f_k|} s$ and $w = \frac{1}{\sqrt{\Lambda|X_k|}} z$, we obtain

$$\frac{1}{2} \int_{\tilde{\Theta}_{k,\sigma}} |\nabla_{\tilde{A}_\varepsilon} \tilde{u}_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |\tilde{u}_\varepsilon|^2)^2 + |\partial_s \tilde{A}_{\varepsilon,z} - \partial_z \tilde{A}_{\varepsilon,s}|^2 \leq C_{k,\Lambda} F_{\varepsilon,k,\sigma}.$$

The co-area formula provides the existence of m_ε with $M_\varepsilon^{-1} \leq m_\varepsilon \leq 2M_\varepsilon^{-1}$ such that setting $W := \{(s, z) \mid |\tilde{u}_\varepsilon(s, z)| \leq 1 - m_\varepsilon\}$ has perimeter (for the Euclidean metric) bounded by $CC_{k,\Lambda} \varepsilon M_\varepsilon^2$. We may then apply this proposition to the configuration $(\tilde{u}_\varepsilon, \tilde{A}_\varepsilon)$, with initial radius $r_0 = CC_{k,\Lambda} \varepsilon M_\varepsilon^2$ and final radius $r_1 = |f_k|^2 |X_k|^2 M_\varepsilon^{-1}$ (provided $\varepsilon \ll 1$ such that $r_1 \leq 1$). This yields a collection of disjoint closed balls $\tilde{B} = \{\tilde{B}_i\}_{i \in I}$ such that

- $r(\tilde{B}) = |f_k|^2 |X_k|^2 M_\varepsilon^{-1}$,
- $\tilde{V} = \tilde{\Theta}_{k,\sigma}^\varepsilon \cap \cup_i \tilde{B}_i$ covers $\{(s, z) \mid |\tilde{u}_\varepsilon(s, z)| - 1| \geq \frac{1}{2}\} \cap \tilde{\Theta}_{k,\sigma}^\varepsilon$,
- and writing $d_i = \deg(v_\varepsilon/|v_\varepsilon|, \partial \tilde{B}_i)$ if $\tilde{B}_i \subset \tilde{\Theta}_{k,\sigma}^\varepsilon$ and $d_i = 0$ otherwise, we have for each i ,

$$\frac{1}{2} \int_{(\tilde{B}_i \cap \Theta_{k,\sigma}^\varepsilon) \setminus W} |\nabla_{\tilde{A}_\varepsilon}(\tilde{u}_\varepsilon/|\tilde{u}_\varepsilon|)|^2 + r_1(r_1 - r_0) |\partial_s A_{\varepsilon,z} - \partial_z A_{\varepsilon,s}|^2 \geq \pi |d_i| \left(\log \frac{r_1}{r_0} - C \right).$$

Moreover, by [SS07, Lemma 3.4], we have

$$\begin{aligned} \int_{(\tilde{B}_i \cap \Theta_{k,\sigma}^\varepsilon) \setminus W} |\nabla_{\tilde{A}_\varepsilon} \tilde{u}_\varepsilon|^2 &\geq \int_{(\tilde{B}_i \cap \Theta_{k,\sigma}^\varepsilon) \setminus W} |\tilde{u}_\varepsilon|^2 |\nabla_{\tilde{A}_\varepsilon}(\tilde{u}_\varepsilon/|\tilde{u}_\varepsilon|)|^2 + |\nabla |\tilde{u}_\varepsilon||^2 \\ &\geq (1 - m_\varepsilon)^2 \int_{(\tilde{B}_i \cap \Theta_{k,\sigma}^\varepsilon) \setminus W} |\nabla_{\tilde{A}_\varepsilon}(\tilde{u}_\varepsilon/|\tilde{u}_\varepsilon|)|^2. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{2} \int_{(\tilde{B}_i \cap \Theta_{k,\sigma}^\varepsilon) \setminus W} |\nabla_{\tilde{A}_\varepsilon} \tilde{u}_\varepsilon|^2 + r_1(r_1 - r_0) |\partial_s \tilde{A}_{\varepsilon,z} - \partial_z \tilde{A}_{\varepsilon,s}|^2 \\ \geq (1 - m_\varepsilon)^2 \pi |d_i| \left(\log \frac{r_1}{r_0} - C \right) \\ \geq \pi |d_i| (|\log \varepsilon| - C \log M_\varepsilon + \log(C_{k,\Lambda}^{-1} |f_k|^2 |X_k|^2)). \end{aligned}$$

In particular, we deduce that

$$(9.3) \quad D = \sum_{i \in I} |d_i| \leq \frac{C_{k,\Lambda} F_{\varepsilon,k,\sigma}}{|\log \varepsilon|}.$$

Then, by changing variables once again, we obtain balls B_i , the images of the \tilde{B}_i 's by the change of variable, which are geodesic balls for the metric g_k and whose sum of radii is bounded by $\max \left\{ \frac{\sqrt{\Lambda}}{|f_k|}, \frac{1}{\sqrt{\Lambda} |X_k|} \right\} r(\tilde{B}) \leq R_{k,\Lambda} |f_k|^2 |X_k|^2 M_\varepsilon^{-1}$. Items (2) and (3) immediately follow from the change of variables and the properties satisfied by \tilde{u}_ε and \tilde{A}_ε .

Item (4) follows from [SS07, Theorem 6.1]. Indeed, denoting \tilde{a}_i the center of \tilde{B}_i and letting $\tilde{\mu}_{\varepsilon,k,\sigma} = 2\pi \sum_i d_i \delta_{\tilde{a}_i}$, this theorem yields, for any $\tilde{\xi} \in C_0^{0,1}(\tilde{\Theta}_{k,\sigma})$, that

$$\left| \int_{\tilde{\Theta}_{k,\sigma}} (\mu(\tilde{u}_\varepsilon, \tilde{A}_\varepsilon) - \tilde{\mu}_{\varepsilon,k,\sigma}) \wedge \tilde{\xi} \right| \leq C \|\tilde{\xi}\|_{C^{0,1}} r(\tilde{B}) C_{k,\Lambda} F_{\varepsilon,k,\sigma},$$

where C is a universal constant, but, by change of variables, we have

$$\int_{\tilde{\Theta}_{k,\sigma}} (\mu(\tilde{u}_\varepsilon, \tilde{A}_\varepsilon) - \tilde{\mu}_{\varepsilon,k,\sigma}) \wedge \tilde{\xi} = \int_{\Theta_{k,\sigma}} \left(J_{\varepsilon,k,\sigma} - 2\pi \sum_{i \in I} d_i \delta_{a_i} \right) \wedge \xi,$$

where a_i is the center of the ball B_i , i.e. the image by the change of variables of \tilde{a}_i , and $\xi(t, w) = \tilde{\xi} \left(\frac{|f_k|}{\sqrt{\Lambda}} t, \sqrt{\Lambda} |X_k| w \right)$. This concludes the proof. \square

Throughout the rest of this section $C > 0$ denotes a universal constant that may change from line to line.

Proof of Theorem 1.3. We proceed similarly to [SS04, Ser17]. We set $\vartheta_{\varepsilon,k,\sigma}$ to be the $\mu_{\varepsilon,k,\sigma}$ of Lemma 9.1 (item 4) if the assumption $F_{\varepsilon,k,\sigma} \leq M_\varepsilon$ is verified, and 0 if not. We note that

$$(9.4) \quad \|J_{\varepsilon,k,\sigma} - \vartheta_{\varepsilon,k,\sigma}\|_{C_0^{0,1}(\Theta_{k,\sigma})^*} \leq C(C_{k,\Lambda}|f_k|^2|X_k|^2 + 1)M_\varepsilon^{-1/2}F_{\varepsilon,k,\sigma}$$

is true in all cases. Indeed, either $F_{\varepsilon,k,\sigma} \leq M_\varepsilon$ in which case the result is true by item 4 of Lemma 9.1 since $M_\varepsilon^{-1/2} \geq M_\varepsilon^{-1}$, or $\vartheta_{\varepsilon,k,\sigma} = 0$ in which case, for any $\xi \in C_0^{0,1}(\Theta_{k,\sigma})$, starting from (9.1) and writing $|\langle \nabla u_\varepsilon - iA_\varepsilon u_\varepsilon, iu_\varepsilon \rangle| \leq |\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon| + |1 - |u_\varepsilon|^2| |\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon|$ (note that $|1 - |u_\varepsilon|^2| \leq |1 - |u_\varepsilon|^2|$), we obtain with the Cauchy-Schwarz inequality, using the boundedness of $\Theta_{k,\sigma}$,

$$\begin{aligned} \left| \int_{\Theta_{k,\sigma}} J_{\varepsilon,k,\sigma} \wedge \xi \right| &\leq \|\nabla \xi\|_{L^\infty} \int_{\Theta_{k,\sigma}} (|\partial_t u_\varepsilon - iu_\varepsilon \Phi_\varepsilon| + |\partial_w u_\varepsilon - iu_\varepsilon B_{\varepsilon,w}|)(1 + |1 - |u_\varepsilon|^2|) \\ &\quad + \|\xi\|_{L^\infty} \int_{\Theta_{k,\sigma}} |\partial_t B_{\varepsilon,w} - \partial_w \Phi_\varepsilon| \\ &\leq C\|\xi\|_{C^{0,1}}(\sqrt{F_{\varepsilon,k,\sigma}} + \varepsilon F_{\varepsilon,k,\sigma}). \end{aligned}$$

Moreover, since $F_{\varepsilon,k,\sigma} \geq M_\varepsilon$, we have $\sqrt{F_{\varepsilon,k,\sigma}} + \varepsilon F_{\varepsilon,k,\sigma} \leq 2M_\varepsilon^{-1/2}F_{\varepsilon,k,\sigma}$ and thus we find that (9.4) holds as well.

We may now write that

$$\begin{aligned} &\int_{\Theta_{k,\sigma}} \chi_k \left(\frac{|f_k|^2}{\Lambda} |\partial_t u_\varepsilon - iu_\varepsilon \Phi_\varepsilon|^2 + \Lambda |X_k|^2 |\partial_w u_\varepsilon - iu_\varepsilon B_{\varepsilon,w}|^2 \right. \\ &\quad \left. + |f_k|^5 |X_k|^5 M_\varepsilon^{-2} |\partial_t B_{\varepsilon,w} - \partial_w \Phi_\varepsilon|^2 \right) \\ &\geq (|\log \varepsilon| - C \log M_\varepsilon + \log(C_{k,\Lambda}^{-1} |f_k|^2 |X_k|^2)) \left| \int_{\Theta_{k,\sigma}} |f_k| |X_k| \chi_k \vartheta_{\varepsilon,k,\sigma} \right| \\ &\quad - CC_{k,\Lambda} R_{k,\Lambda}^2 |f_k|^3 |X_k|^3 M_\varepsilon^{-3/4} F_{\varepsilon,k,\sigma}. \end{aligned}$$

Indeed, if we are in a slice where $\vartheta_{\varepsilon,k,\sigma} = 0$, this is trivially true. If not, we apply (9.2) and obtain

$$\begin{aligned} &\int_{(\cup_i B_i) \cap \Theta_{k,\sigma}^\varepsilon} \chi_k \left(\frac{|f_k|^2}{\Lambda} |\partial_t u_\varepsilon - iu_\varepsilon \Phi_\varepsilon|^2 + \Lambda |X_k|^2 |\partial_w u_\varepsilon - iu_\varepsilon B_{\varepsilon,w}|^2 \right. \\ &\quad \left. + |f_k|^5 |X_k|^5 M_\varepsilon^{-2} |\partial_t B_{\varepsilon,w} - \partial_w \Phi_\varepsilon|^2 \right) \\ &\geq 2\pi \sum_{i \in I} |d_i| \min_{B_i} \chi_k (|\log \varepsilon| - C \log M_\varepsilon + \log(C_{k,\Lambda}^{-1} |f_k|^2 |X_k|^2)) |f_k| |X_k|. \end{aligned}$$

Besides, we have $\min_{B_i} \chi_k \geq \chi_k(a_i) - C \max \left\{ \frac{\sqrt{\Lambda}}{|f_k|}, \frac{1}{\sqrt{\Lambda}|X_k|} \right\} r_i \|\chi_k\|_{C^{0,1}}$. Plugging this in, using item 1 of Lemma 9.1 and (9.3), yields the desired inequality.

Combining with (9.4), we find

$$\begin{aligned} & \int_{\Theta_{k,\sigma}} \chi_k \left(\frac{|f_k|^2}{\Lambda} |\partial_t u_\varepsilon - i u_\varepsilon \Phi_\varepsilon|^2 + \Lambda |X_k|^2 |\partial_w u_\varepsilon - i u_\varepsilon B_{\varepsilon,w}|^2 \right. \\ & \quad \left. + |f_k|^5 |X_k|^5 M_\varepsilon^{-2} |\partial_t B_{\varepsilon,w} - \partial_w \Phi_\varepsilon|^2 \right) \\ & \geq (|\log \varepsilon| - C \log M_\varepsilon + \log(C_{k,\Lambda}^{-1} |f_k|^2 |X_k|^2)) \left| \int_{\Theta_{k,\sigma}} |f_k| |X_k| \chi_k J_{\varepsilon,k,\sigma} \right| \\ & \quad - C \tilde{C}_{k,\Lambda} \tilde{R}_{k,\Lambda}^2 M_\varepsilon^{-3/4} F_{\varepsilon,k,\sigma} - C(\tilde{C}_{k,\Lambda} + 1) |f_k| |X_k| |\log \varepsilon| M_\varepsilon^{-1/2} F_{\varepsilon,k,\sigma}, \end{aligned}$$

where $\tilde{C}_{k,\Lambda} := C_{k,\Lambda} |f_k| |X_k|$ and $\tilde{R}_{k,\Lambda} := R_{k,\Lambda} |f_k| |X_k|$.

Let us observe that

$$\log(C_{k,\Lambda}^{-1} |f_k|^2 |X_k|^2) \geq \log \min(\Lambda, \Lambda^{-1}) + \log \min(C_{k,1}^{-1} |f_k|^2 |X_k|^2, 1).$$

Moreover (see (9.5))

$$\begin{aligned} & \log \min(C_{k,1}^{-1} |f_k|^2 |X_k|^2, 1) \left| \int_{\Theta_{k,\sigma}} |f_k| |X_k| \chi_k J_{\varepsilon,k,\sigma} \right| \\ & \geq -|\log \min\{C_{k,1}^{-1} |f_k|^2 |X_k|^2, 1\}| |f_k| |X_k| \int_{\Theta_{k,\sigma}} \chi_k |V_\varepsilon| \\ & \geq -C \max_{s \in (0,1)} s |\log s| \int_{\Theta_{k,\sigma}} \chi_k |V_\varepsilon| \\ & = -C \int_{\Theta_{k,\sigma}} \chi_k |V_\varepsilon|. \end{aligned}$$

Notice that, since we assumed that X_k is along the direction e_1 , we have

$$(9.5) \quad J_{\varepsilon,k,\sigma} = J(e_t, e_1) = V_\varepsilon \cdot e_2 = V_\varepsilon \cdot \frac{X_k^\perp}{|X_k|},$$

so we may bound

$$\left| \int_{\Theta_{k,\sigma}} |f_k| |X_k| \chi_k J_{\varepsilon,k,\sigma} \right| \geq \left| \int_{\Theta_{k,\sigma}} \chi_k f_k V_\varepsilon \cdot X_k^\perp \right|.$$

We also observe that

$$\int_{\Theta_{k,\sigma}} \chi_k |X_k|^2 |\partial_w u_\varepsilon - i u_\varepsilon B_{\varepsilon,w}|^2 = \int_{\Theta_{k,\sigma}} \chi_k |X_k \cdot (\nabla u_\varepsilon - i u_\varepsilon B_\varepsilon)|^2.$$

Plugging in and integrating with respect to σ yields

$$\begin{aligned} & \int_{D_k} \chi_k \left(\frac{|f_k|^2}{\Lambda} |\partial_t u_\varepsilon - i u_\varepsilon \Phi_\varepsilon|^2 + \Lambda |X_k \cdot (\nabla u_\varepsilon - i u_\varepsilon B_\varepsilon)|^2 \right. \\ & \quad \left. + |f_k|^5 |X_k|^5 M_\varepsilon^{-2} |\partial_t B_{\varepsilon,w} - \partial_w \Phi_\varepsilon|^2 \right) \\ & \geq (|\log \varepsilon| - C \log M_\varepsilon - \log \max(\Lambda, \Lambda^{-1})) \left| \int_{D_k} \chi_k f_k V_\varepsilon \cdot X_k^\perp \right| - C \int_{D_k} \chi_k |V_\varepsilon| \\ & \quad - C \tilde{C}_{k,\Lambda} \tilde{R}_{k,\Lambda}^2 M_\varepsilon^{-3/4} F_{\varepsilon,k} - C(\tilde{C}_{k,\Lambda} + 1) |f_k| |X_k| |\log \varepsilon| M_\varepsilon^{-1/2} F_{\varepsilon,k}, \end{aligned}$$

where $F_{\varepsilon,k} := \int F_{\varepsilon,k,\sigma} d\sigma$. It is important to note that since $\tilde{C}_{k,\Lambda}, \tilde{R}_{k,\Lambda} \leq C$ for some constant C depending only on $\Lambda, \|f\|_\infty$, and $\|X\|_\infty$, this holds as well if $f_k = 0$ or $X_k = 0$.

We may next replace X_k by X and f_k by f in the first two terms of the left-hand side and the $\int_{D_k} \chi_k f_k V_\varepsilon \cdot X_k^\perp$ term, and using that $|X - X_k| \leq C M_\varepsilon^{-1/4} \|X\|_{C^{0,1}}$ and $|f - f_k| \leq C M_\varepsilon^{-1/4} \|f\|_{C^{0,1}}$, the error thus created is bounded above by

$$CC(\Lambda)C_0(f, X)C_1(f, X)M_\varepsilon^{-1/4} |\log \varepsilon| F_\varepsilon(D_k),$$

where $C_0(f, X) := \max\{\|f\|_\infty, \|X\|_\infty, 1\}$, $C_1(f, X) := \max\{\|f\|_{C^{0,1}}, \|X\|_{C^{0,1}}, 1\}$, $C(\Lambda) = \max\{\frac{1}{\Lambda}, \Lambda\}$, and

$$F_\varepsilon(D_k) := \frac{1}{2} \int_{D_k} |\partial_t u_\varepsilon - i u_\varepsilon \Phi_\varepsilon|^2 + |\nabla u_\varepsilon - i u_\varepsilon B_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 + |\operatorname{curl} A_\varepsilon|^2.$$

Here we have used that by definition of V_ε , we have $\int_{D_k} |V_\varepsilon| \leq C F_{\varepsilon,k}(D_k)$. Let us note that

$$|f_k|^5 |X_k|^5 M_\varepsilon^{-2} |\partial_t B_{\varepsilon,w} - \partial_w \Phi_\varepsilon|^2 \leq C_0(f, X)^{10} M_\varepsilon^{-2} \int_{D_k} |\operatorname{curl} A_\varepsilon|^2 \leq 2C_0(f, X)^{10} M_\varepsilon^{-2} F_\varepsilon(D_k),$$

$\tilde{C}_{k,\Lambda} \leq C(\Lambda)C_0(f, X)^6$, and $\tilde{R}_{k,\Lambda}^2 \leq C(\Lambda)C_0(f, X)^6$. Summing over k , using that $\sum_k \chi_k = 1$ in K (the union of the supports of f and X) and the finite overlap of the covering, we are led to

$$\begin{aligned} & \int_{(0,T) \times \omega} \frac{|f|^2}{\Lambda} |\partial_t u_\varepsilon - i u_\varepsilon \Phi_\varepsilon|^2 + \Lambda |X \cdot (\nabla u_\varepsilon - i u_\varepsilon B_\varepsilon)|^2 \\ & \geq (|\log \varepsilon| - C \log M_\varepsilon - \log \max(\Lambda, \Lambda^{-1})) \left| \int_{(0,T) \times \omega} f V_\varepsilon \cdot X^\perp \right| \\ & \quad - C \int_{(0,T) \times \omega} \sum_k \chi_k |V_\varepsilon| - \operatorname{Error}(\varepsilon, \Lambda, f, X), \end{aligned}$$

where

$$\begin{aligned} \text{Error}(\varepsilon, \Lambda, f, X) = CF_\varepsilon(u_\varepsilon, A_\varepsilon) & \left(C(\Lambda)C_0(f, X)C_1(f, X)|\log \varepsilon|M_\varepsilon^{-1/4} + C_0(f, X)^{10}M_\varepsilon^{-2} \right. \\ & \left. + C^2(\Lambda)C_0(f, X)^{12}M_\varepsilon^{-3/4} + C(\Lambda)C_0(f, X)^8|\log \varepsilon|M_\varepsilon^{-1/2} \right). \end{aligned}$$

We may now use Theorem 1.2 (notice that X , f , and χ_k are compactly supported in $\omega \times (0, T)$) to replace the velocity by its polyhedral approximation. Since we assume that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq M|\log \varepsilon|^m$, Theorem 1.2 with $n > \frac{1}{3}(2 - m)$ implies that

$$\int_{(0,T) \times \omega} fV_\varepsilon \cdot X^\perp = \int_{(0,T) \times \omega} f\nu_\varepsilon \wedge (-X_2 dx_1 + X_1 dx_2) + O\left(|\log \varepsilon|^{-\frac{1}{2}(m+3n)}\right)$$

and

$$\int_{(0,T) \times \omega} \sum_k \chi_k |V_\varepsilon| \leq C \int_{(0,T) \times \omega} \max(|\nu_\varepsilon \wedge dx_1|, |\nu_\varepsilon \wedge dx_2|) + O\left(|\log \varepsilon|^{-\frac{1}{2}(m+3n)}\right),$$

which gives

$$\begin{aligned} & \int_{(0,T) \times \omega} \frac{|f|^2}{\Lambda} |\partial_t u_\varepsilon - iu_\varepsilon \Phi_\varepsilon|^2 + \Lambda |X \cdot (\nabla u_\varepsilon - iu_\varepsilon B_\varepsilon)|^2 \\ & \geq (|\log \varepsilon| - C \log M_\varepsilon - \log \max(\Lambda, \Lambda^{-1})) \left| \int_{(0,T) \times \omega} f\nu_\varepsilon \wedge (-X_2 dx_1 + X_1 dx_2) \right| \\ & - C \int_{(0,T) \times \omega} \max(|\nu_\varepsilon \wedge dx_1|, |\nu_\varepsilon \wedge dx_2|) + O\left(|\log \varepsilon|^{-\frac{1}{2}(m+3n)+1}\right) - \text{Error}(\varepsilon, \Lambda, f, X). \end{aligned}$$

We observe that $-\frac{1}{2}(m + 3n) + 1 < 0$ and therefore the error associated to our approximation goes to zero as $\varepsilon \rightarrow 0$. In addition, for ε small enough depending on Λ , f , and X , we have

$$\max(\Lambda, \Lambda^{-1}) \leq M_\varepsilon \quad \text{and} \quad \text{Error}(\varepsilon, \Lambda, f, X) = O(|\log \varepsilon|^{-\frac{1}{2}(m+3n)+1}).$$

This concludes the proof. \square

APPENDIX A. SMOOTH APPROXIMATION OF THE FUNCTION ζ

In this section we present the proof of Proposition 5.1. We will regularize the function ζ by convolution. Having into account that we need to provide a quantitative estimate of the second fundamental form of the approximation, we will first displace a bit the points of the collection. The reason for this is that the displacement will ensure that the vectors $\nu_{(i,j)}$ (see Definition 5.1) will satisfy “good” angle conditions between each other. This will permit us to characterize the set where the gradient of the convolution is small, which will translate into a control on its second fundamental form.

A.1. Displacement of the points. First, we prove some basic geometric properties which allow us to perform the displacement of the points (see Proposition A.1).

Lemma A.1. *Consider a unit vector $\nu \in \mathbb{R}^3$, a ball $B \subset \mathbb{R}^3$ of diameter $D > 0$, and a point $x \in B$. There exists a universal constant $\vartheta_1 > 0$ such that, for any $0 < \vartheta \leq \vartheta_1$, letting $C_{\nu,x,D,\vartheta}$ denote the infinite cylinder whose axis is parallel to ν and passes through x , and whose radius is $2D\vartheta$, we have*

$$\left| \nu \times \frac{x-y}{|x-y|} \right| \geq \vartheta$$

for any $y \in B \setminus C_{\nu,x,D,\vartheta}$.

Proof. Let $y \in B \setminus C_{\nu,x,D,\vartheta}$. We recall that

$$\left| \nu \times \frac{x-y}{|x-y|} \right| = |\nu| \left| \frac{x-y}{|x-y|} \right| |\sin \theta| = |\sin \theta|,$$

where θ is the angle formed by ν and $\frac{x-y}{|x-y|}$. Since the radius of the cylinder $C_{\nu,x,D,\vartheta}$ is $2D\vartheta$ and $y \notin C_{\nu,x,D,\vartheta}$, from Pythagoras' theorem we deduce that

$$\sqrt{D^2 + (2D\vartheta)^2} |\sin \theta| \geq 2D\vartheta.$$

Therefore, for any $0 < \vartheta \leq \vartheta_1$, with ϑ_1 such that $2/\sqrt{1 + (2\vartheta_1)^2} = 1$, we have

$$|\sin \theta| \geq \vartheta.$$

This concludes the proof. \square

Lemma A.2. *Consider two unit vectors $\nu_1, \nu_2 \in \mathbb{R}^3$ with $\nu_1 \neq \pm\nu_2$, a ball $B \subset \mathbb{R}^3$ of diameter $D > 0$, and a point $x \in B$. Let P be the plane perpendicular to $\nu_1 \times \nu_2$ and that passes through x . For any $0 < \vartheta \leq \vartheta_1$, where ϑ_1 is as in Lemma A.1, letting*

$$P_{\nu_1,\nu_2,x,D,\vartheta} = \{y \in \mathbb{R}^3 \mid d(y, P) \leq 2D\vartheta\},$$

if $|\nu_1 \times \nu_2| \geq \vartheta$ then

$$\left| \det \left(\nu_1, \nu_2, \frac{x-y}{|x-y|} \right) \right| \geq \vartheta^2$$

for any $y \in B \setminus P_{\nu_1,\nu_2,x,D,\vartheta}$.

Proof. Let $y \in B \setminus P_{\nu_1,\nu_2,x,D,\vartheta}$. We recall that

$$\left| \det \left(\nu_1, \nu_2, \frac{x-y}{|x-y|} \right) \right| = \left| (\nu_1 \times \nu_2) \cdot \frac{x-y}{|x-y|} \right| = |\nu_1 \times \nu_2| \left| \frac{x-y}{|x-y|} \right| |\cos \theta| \geq \vartheta |\cos \theta|,$$

where θ is the angle formed by $\nu_1 \times \nu_2$ and $\frac{x-y}{|x-y|}$. Since $y \in B$ and $d(y, P) \geq 2D\vartheta$, from Pythagoras' theorem we deduce that

$$\sqrt{D^2 + (2D\vartheta)^2} |\cos \theta| \geq 2D\vartheta.$$

Therefore, for any $0 < \vartheta \leq \vartheta_1$, we have $|\cos \theta| \geq \vartheta$, which implies

$$\left| \det \left(\nu_1, \nu_2, \frac{x-y}{|x-y|} \right) \right| \geq \vartheta^2.$$

□

The previous lemmas allow us to prove the following result.

Proposition A.1. *Let $\mathcal{A} = \{a_1, \dots, a_m\}$ be a collection of m not necessarily distinct points. Define $D_{\mathcal{A}} := \max_{1 \leq i < j \leq m} |a_i - a_j|$ to be the maximum Euclidean distance between any of the points of \mathcal{A} and assume that $D_{\mathcal{A}} > 0$. Then there exist universal constants $C_1, C > 0$ and a collection of points $\mathcal{A}' = \{b_1, \dots, b_m\}$ such that, for any $0 < \vartheta \leq \min\{C_1 m^{-5}, \vartheta_1\}$, where ϑ_1 is as in Lemmas A.1 and A.2, the following hold:*

- (1) $b_i \neq b_j$ for any $i \neq j$.
- (2) Define

$$\nu_{(i,j)} := \frac{b_i - b_j}{|b_i - b_j|} \quad \text{for } (i, j) \in \Lambda_m := \{(p, q) \mid 1 \leq p < q \leq m\}.$$

Then for any $\alpha, \beta, \gamma \in \Lambda_m$ with $\alpha \neq \beta \neq \gamma$, we have

$$|\nu_{\alpha} \times \nu_{\beta}| \geq \vartheta \quad \text{and} \quad |\det(\nu_{\alpha}, \nu_{\beta}, \nu_{\gamma})| \geq \vartheta^2.$$

- (3) $|a_l - b_l| \leq CD_{\mathcal{A}} l^5 \vartheta$ for any $l \in \{1, \dots, m\}$.

Proof. Let us first observe that, by definition of $D_{\mathcal{A}}$, there exists a closed ball $B_{\mathcal{A}}$ of diameter $D_{\mathcal{A}}$ which contains all the points of the collection \mathcal{A} .

We proceed by induction. Define $b_1 = a_1$ and let $0 < \vartheta \leq \min\{C_1 m^{-5}, \vartheta_1\}$, where C_1 is some universal constant to be specified later. Assume that we have defined a collection $\{b_1, \dots, b_l\} \subset B_{\mathcal{A}}$ with $1 < l < m$ such that:

- For any $\alpha, \beta, \gamma \in \Lambda_l$ with $\alpha \neq \beta \neq \gamma$, we have

$$|\nu_{\alpha} \times \nu_{\beta}| \geq \vartheta \quad \text{and} \quad |\det(\nu_{\alpha}, \nu_{\beta}, \nu_{\gamma})| \geq \vartheta^2.$$

- $|a_i - b_i| \leq 8D_{\mathcal{A}}(i-1)^5 \vartheta$ for any $i \in \{1, \dots, l\}$.

We will next find $b_{l+1} \in B_{\mathcal{A}}$ such that the collection $\{b_1, \dots, b_l, b_{l+1}\} \subset B_{\mathcal{A}}$ satisfies that:

- For any $\alpha, \beta, \gamma \in \Lambda_{l+1}$ with $\alpha \neq \beta \neq \gamma$, we have

$$|\nu_{\alpha} \times \nu_{\beta}| \geq \vartheta \quad \text{and} \quad |\det(\nu_{\alpha}, \nu_{\beta}, \nu_{\gamma})| \geq \vartheta^2.$$

- $|a_{l+1} - b_{l+1}| \leq 8D_{\mathcal{A}} l^5 \vartheta$.

For any $i \in \{1, \dots, l\}$ and $\alpha \in \Lambda_l$, let $C_{\nu_{\alpha}, b_i, D_{\mathcal{A}}, \vartheta}$ denote the cylinder defined in Lemma A.1 with $\nu = \nu_{\alpha}$, $B = B_{\mathcal{A}}$, $D = D_{\mathcal{A}}$, and $x = b_i$. In addition, for any

$i \in \{1, \dots, l\}$ and $\alpha, \beta \in \Lambda_l$ with $\alpha \neq \beta$, let $P_{\nu_\alpha, \nu_\beta, b_i, D_{\mathcal{A}}, \vartheta}$ denote the set defined in Lemma A.2 with $\nu_1 = \nu_\alpha$, $\nu_2 = \nu_\beta$, $B = B_{\mathcal{A}}$, $D = D_{\mathcal{A}}$, and $x = b_i$. We define

$$X_{\text{bad}}^l := \left(\bigcup_{i \in \{1, \dots, l\}} \bigcup_{\alpha \in \Lambda_l} C_{\nu_\alpha, b_i, D_{\mathcal{A}}, \vartheta} \right) \cup \left(\bigcup_{i \in \{1, \dots, l\}} \bigcup_{\alpha, \beta \in \Lambda_l, \alpha \neq \beta} P_{\nu_\alpha, \nu_\beta, b_i, D_{\mathcal{A}}, \vartheta} \right).$$

By Lemmas A.1 and A.2, we conclude that, for any $y \in B_{\mathcal{A}} \setminus X_{\text{bad}}^l$, the collection of points $\{b_1, \dots, b_l, y\}$ is such that, for any $\alpha, \beta, \gamma \in \Lambda_{l+1}$ with $\alpha \neq \beta \neq \gamma$,

$$|\nu_\alpha \times \nu_\beta| \geq \vartheta \quad \text{and} \quad |\det(\nu_\alpha, \nu_\beta, \nu_\gamma)| \geq \vartheta^2.$$

Let us observe that

$$|X_{\text{bad}}^l| \leq C(l|\Lambda_l|D_{\mathcal{A}}^3\vartheta + l|\Lambda_l|^2D_{\mathcal{A}}^3\vartheta) \leq Cl^5D_{\mathcal{A}}^3\vartheta \leq Cm^5\vartheta|B_{\mathcal{A}}|,$$

where C is a universal constant. In particular, for some sufficiently small universal constant $C_1 > 0$, we have

$$|B_{\mathcal{A}} \setminus X_{\text{bad}}^l| \leq \frac{1}{10}|B_{\mathcal{A}}|$$

for any $0 < \vartheta \leq \min\{C_1m^{-5}, \vartheta_1\}$. By recalling the definition of the radius (resp. thickness) of $C_{\nu_\alpha, b_i, D_{\mathcal{A}}, \vartheta}$ (resp. $P_{\nu_\alpha, \nu_\beta, b_i, D_{\mathcal{A}}, \vartheta}$) and since $a_{l+1} \in B_{\mathcal{A}}$, we deduce that there exists a point $b_{l+1} \in B_{\mathcal{A}} \setminus X_{\text{bad}}^l$ such that

$$|a_{l+1} - b_{l+1}| \leq 2(2D_{\mathcal{A}}\vartheta l^3 + 2D_{\mathcal{A}}\vartheta l^5) \leq 8D_{\mathcal{A}}\vartheta l^5.$$

Therefore the collection $\{b_1, \dots, b_l, b_{l+1}\}$ satisfies the desired properties. This concludes the proof of the induction step and the proposition is thus proved. \square

A.2. Proof of Proposition 5.1.

Proof. Let us begin by defining the function ζ_λ . Let $\mathcal{A}' = \{p'_1, \dots, p'_k, n'_1, \dots, n'_k\}$ be the collection of points given by Proposition A.1 with $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$. Observe that, for any $0 < \vartheta \leq C_0(2k)^{-5}$, where $C_0 = \min(C_1, \vartheta_1)$, we have

$$L(\mathcal{A}') \leq \sum_{i=1}^k |p'_i - n'_i| \leq \sum_{i=1}^k |p_i - n_i| + |p_i - p'_i| + |n_i - n'_i| \leq L(\mathcal{A}) + CD_{\mathcal{A}}(2k)^6\vartheta.$$

An analogous argument shows that $L(\mathcal{A}) \leq L(\mathcal{A}') + CD_{\mathcal{A}}(2k)^6\vartheta$. Therefore

$$(A.1) \quad |L(\mathcal{A}) - L(\mathcal{A}')| \leq CD_{\mathcal{A}}(2k)^6\vartheta,$$

where throughout the proof $C > 0$ denotes a universal constant that may change from line to line. Remember that by Lemma 5.1 there exists a 1-Lipschitz function $\zeta^* : \cup_{i=1, \dots, k} \{p'_i, n'_i\} \rightarrow \mathbb{R}$ such that

$$L(\mathcal{A}') = \sum_{i=1}^k \zeta^*(p'_i) - \zeta^*(n'_i).$$

Define the function ζ as in Definition 5.1, i.e. set

$$\zeta(x) := \max_{i \in \{1, \dots, k\}} \left(\zeta^*(p'_i) - \max_{j \in \{1, \dots, 2k\}} d_{(i,j)}(x) \right),$$

where

$$d_{(i,j)}(x) := \langle p'_i - x, \nu_{(i,j)} \rangle, \quad \nu_{(i,j)} = \begin{cases} \frac{p'_i - a'_j}{|p'_i - a'_j|} & \text{if } p'_i \neq a'_j \\ 0 & \text{if } p'_i = a'_j \end{cases}.$$

Lemma 5.2 yields that $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a 1-Lipschitz function such that

$$\sum_{i=1}^k \zeta(p'_i) - \zeta(n'_i) = \sum_{i=1}^k \zeta^*(p'_i) - \zeta^*(n'_i) = L(\mathcal{A}').$$

Next, we regularize the function ζ . Let $\varphi \in C_c^\infty(B(0, 1), \mathbb{R}_+)$ be a mollifier such that $\int_{\mathbb{R}^3} \varphi(x) dx = 1$. Letting

$$(A.2) \quad \lambda := \vartheta^{1/\rho} \quad \text{for } \rho \in (0, 1/2),$$

we define

$$\zeta_\lambda(\cdot) := \varphi_\lambda * \zeta(\cdot) = \int_{\mathbb{R}^3} \varphi_\lambda(\cdot - y) \zeta(y) dy \quad \text{with } \varphi_\lambda(\cdot) = \frac{1}{\lambda} \varphi\left(\frac{\cdot}{\lambda}\right).$$

Argument for the first statement. Observe that $\|\zeta - \zeta_\lambda\|_{L^\infty(\mathbb{R}^3)} \leq \lambda$ from which we deduce that

$$(A.3) \quad \left| L(\mathcal{A}') - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq 2k\lambda.$$

By combining (A.1) with (A.3), we obtain

$$(A.4) \quad \left| L(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq CD_{\mathcal{A}}(2k)^6 \vartheta + 2k\lambda.$$

On the other hand, note that

$$\left| \sum_{i=1}^k \zeta(p_i) - \zeta(n_i) - \sum_{i=1}^k \zeta(p'_i) - \zeta(n'_i) \right| \leq \sum_{i=1}^k |p_i - n_i| + |p'_i - n'_i| \leq CD_{\mathcal{A}}(2k)^6 \vartheta.$$

By combining the previous estimate with (A.3), we get

$$(A.5) \quad \left| \sum_{i=1}^k \zeta(p_i) - \zeta(n_i) - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq CD_{\mathcal{A}}(2k)^6 \vartheta + 2k\lambda.$$

Then from (A.4) and (A.5), we deduce that

$$\left| L(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p_i) - \zeta_\lambda(n_i) \right| \leq CD_{\mathcal{A}}(2k)^6 \lambda^\rho.$$

Argument for the second statement. Note that

$$(A.6) \quad \nabla\zeta_\lambda(x) = \int_{B(x,\lambda)} \varphi_\lambda(x-y) \nabla\zeta(y) dy$$

for any $x \in \mathbb{R}^3$. We define

$$\Lambda := \{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq 2k, i \neq j\}.$$

Then, letting

$$\zeta_{(i,j)}(\cdot) := \zeta(p'_i) - d_{(i,j)}(\cdot) \quad \text{for } (i, j) \in \Lambda,$$

observe that, for almost every $y \in \mathbb{R}^3$,

$$\nabla\zeta(y) = \nu_{(i,j)} \quad \text{if } \zeta(y) = \zeta_{(i,j)}(y) \text{ for some } (i, j) \in \Lambda.$$

Since $|\nu_{(i,j)}| = 1$ for any $(i, j) \in \Lambda$, we have

$$|\nabla\zeta_\lambda(x)| \leq \int_{B(x,\lambda)} \varphi_\lambda(x-y) |\nabla\zeta(y)| dy \leq \int_{B(x,\lambda)} \varphi_\lambda(x-y) dy = 1$$

for any $x \in \mathbb{R}^3$.

Argument for the third statement. Let us first prove that there exists a set $P_\lambda \subset \mathbb{R}^3$ such that $|\zeta_\lambda(P_\lambda)| \leq 2\lambda k^2$ and that, for any $0 < \kappa < \vartheta^2/3$,

$$C_\kappa := \{x \in \mathbb{R}^3 \mid |\nabla\zeta_\lambda(x)| < \kappa\} \setminus P_\lambda$$

can be covered by \mathfrak{B}_κ , a collection of at most $(2k)^8$ balls of radius $C\lambda/(\vartheta^2 - 3\kappa)$. We will then conclude the desired result from this.

We start by defining the set P_λ . Let us consider indices $i, j \in \{1, \dots, k\}$ with $i \neq j$. We let

$$P_{i,j} := \{y \in \mathbb{R}^3 \mid \zeta_{(i,j)}(y) = \zeta_{(j,i)}(y)\}$$

and observe that

$$P_{i,j} = \{\zeta(p'_i) - \zeta(p'_j) - \langle p'_i + p'_j - 2y, \nu_{(i,j)} \rangle = 0\}.$$

A simple computation shows that

$$\langle y_1 - y_2, \nu_{(i,j)} \rangle = 0$$

for any $y_1, y_2 \in P_{i,j}$ with $y_1 \neq y_2$. This implies that $P_{i,j}$ is a plane whose normal is $\nu_{(i,j)}$ and therefore

$$\zeta_{(i,j)}(y) = \zeta_{(j,i)}(y) = \frac{\zeta^*(p'_i) + \zeta^*(p'_j) - \langle p'_i + p'_j, \nu_{(i,j)} \rangle}{2}$$

for any $y \in P_{i,j}$. We define

$$P_\lambda := \{y \in \mathbb{R}^3 \mid d(y, P) \leq 2\lambda\}, \quad \text{where } P := \cup_{1 \leq i < j \leq k} P_{i,j}.$$

We immediately check that $|\zeta_\lambda(P_\lambda)| \leq C\lambda k^2$.

Let us now give a characterization of the gradient of ζ_λ . From (A.6), we deduce that

$$\nabla\zeta_\lambda(x) = \sum_{\alpha \in \Lambda} \sigma_\alpha \nu_\alpha, \quad \text{where } \sigma_\alpha = \int_{B(x,\lambda)} \varphi_\lambda(x-y) \mathbf{1}_{\{\zeta(y)=\zeta_\alpha(y)\}} dy \quad \text{for } \alpha \in \Lambda.$$

Observe that $\sigma_\alpha \in [0, 1]$ and $\sum_{\alpha \in \Lambda} \sigma_\alpha = 1$. We conclude that, for any $x \in \mathbb{R}^3$, $\nabla\zeta_\lambda(x)$ can be written as a convex combination of the vectors ν_α 's, $\alpha \in \Lambda$. By Caratheodory's theorem, we deduce that $\nabla\zeta_\lambda(x)$ can be written as a convex combination of at most four of them.

Now consider a number $0 < \vartheta < \vartheta^2/3$ and a point $x \in C_\kappa$. We observe that, since $x \notin P_\lambda$, if there exists a point $y \in \overline{B(x, \lambda)}$ and indices $i, j \in \{1, \dots, k\}$ with $i \neq j$ such that

$$\zeta(y) = \zeta_{(i,j)}(y)$$

then, for any $z \in \overline{B(x, \lambda)}$,

$$\zeta(z) \neq \zeta_{(j,i)}(z).$$

This implies that $\nabla\zeta_\lambda(x)$ can be written as a convex combination of at most four vectors, where if one of them happens to be $\nu_{(i,j)}$ for some $i, j \in \{1, \dots, k\}$ with $i \neq j$ then all the other vectors are different from $\nu_{(j,i)} = -\nu_{(i,j)}$. Recalling that the points of the collection \mathcal{A}' are such that

$$|\nu_\alpha \times \nu_\beta| > \vartheta \quad \text{and} \quad |\det(\nu_\alpha, \nu_\beta, \nu_\gamma)| > \vartheta^2$$

for any $\alpha, \beta, \gamma \in \{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq 2k, i < j\} \subsetneq \Lambda$ with $\alpha \neq \beta \neq \gamma$, we deduce that $\nabla\zeta_\lambda(x)$ can be written as a convex combination of at most four vectors that satisfy these properties.

Let us now show that $\nabla\zeta_\lambda(x)$ cannot be written as a convex combination of three or fewer of the vectors ν_α 's, $\alpha \in \Lambda$. We have three cases to consider:

- If there exists $\alpha \in \Lambda$ such that $\nabla(x)\zeta_\lambda = \nu_\alpha$ then

$$|\nabla\zeta_\lambda(x)| = |\nu_\alpha| = 1.$$

- If there exist $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ such that

$$\nabla\zeta_\lambda(x) = \sigma\nu_\alpha + (1 - \sigma)\nu_\beta$$

for some $\sigma \in (0, 1)$, then

$$\begin{aligned} |\nabla\zeta_\lambda(x)| &\geq \max\{|\nabla\zeta_\lambda(x) \times \nu_\alpha|, |\nabla\zeta_\lambda(x) \times \nu_\beta|\} \\ &= \max\{(1 - \sigma)|\nu_\alpha \times \nu_\beta|, \sigma|\nu_\alpha \times \nu_\beta|\} \\ &\geq \max\{\sigma, 1 - \sigma\}\vartheta \geq \frac{\vartheta}{2}. \end{aligned}$$

- If there exist $\alpha, \beta, \gamma \in \Lambda$ with $\alpha \neq \beta \neq \gamma$ such that

$$\nabla\zeta_\lambda(x) = \sigma_\alpha\nu_\alpha + \sigma_\beta\nu_\beta + \sigma_\gamma\nu_\gamma$$

for some numbers $\sigma_\alpha, \sigma_\beta, \sigma_\gamma \in (0, 1)$ with $\sigma_\alpha + \sigma_\beta + \sigma_\gamma = 1$, then, assuming without loss of generality that $\sigma_\alpha \geq \frac{1}{3}$, we have

$$|\nabla\zeta_\lambda(x)| \geq \sigma_\alpha |\nu_\alpha \cdot (\nu_\beta \times \nu_\gamma)| = \sigma_\alpha |\det(\nu_\alpha, \nu_\beta, \nu_\gamma)| \geq \frac{\vartheta^2}{3}.$$

Since $x \in C_\kappa$ and $\kappa < \frac{\vartheta^2}{3}$ we deduce that the three cases considered above cannot occur. Therefore we conclude that there exist $\alpha, \beta, \gamma, \eta \in \Lambda$ with $\alpha \neq \beta \neq \gamma \neq \eta$ such that

$$\nabla\zeta_\lambda(x) = \sigma_\alpha\nu_\alpha + \sigma_\beta\nu_\beta + \sigma_\gamma\nu_\gamma + \sigma_\eta\nu_\eta$$

for some $\sigma_\alpha, \sigma_\beta, \sigma_\gamma, \sigma_\eta \in (0, 1)$ with $\sigma_\alpha + \sigma_\beta + \sigma_\gamma + \sigma_\eta = 1$.

Let us consider the system of equations

$$(A.7) \quad \zeta_\alpha(y) = \zeta_\beta(y) = \zeta_\gamma(y) = \zeta_\eta(y).$$

We claim that this system admits a unique solution $y \in \mathbb{R}^3$ which in addition satisfies

$$|x - y| \leq \frac{C\lambda}{(\vartheta^2 - 3\kappa)}.$$

Writing $\tilde{y} = y - x$, we observe that \tilde{y} satisfies the linear system of equations $A\tilde{y} = B$, where

$$A = \begin{pmatrix} \nu_\alpha - \nu_\beta \\ \nu_\gamma - \nu_\beta \\ \nu_\eta - \nu_\beta \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \zeta_\alpha(x) - \zeta_\beta(x) \\ \zeta_\gamma(x) - \zeta_\beta(x) \\ \zeta_\eta(x) - \zeta_\beta(x) \end{pmatrix}.$$

Let us check that $|\det(A)| \geq 4(\vartheta^2 - 3\kappa)$. Note that without loss of generality we can assume that $\sigma_\alpha \leq \frac{1}{4}$. Observe that

$$\nabla\zeta_\lambda(x) - \nu_\beta = \sigma_\alpha(\nu_\alpha - \nu_\beta) + \sigma_\gamma(\nu_\gamma - \nu_\beta) + \sigma_\eta(\nu_\eta - \nu_\beta).$$

By Cramer's rule, we have

$$\sigma_\alpha = \frac{\det(\nabla\zeta_\lambda(x) - \nu_\beta, \nu_\gamma - \nu_\beta, \nu_\eta - \nu_\beta)}{\det(\nu_\alpha - \nu_\beta, \nu_\gamma - \nu_\beta, \nu_\eta - \nu_\beta)}.$$

Simple computations show that

$$\det(\nabla\zeta_\lambda(x) - \nu_\beta, \nu_\gamma - \nu_\beta, \nu_\eta - \nu_\beta) = -\det(\nu_\beta, \nu_\gamma, \nu_\eta) + f(\nabla\zeta_\lambda(x)),$$

where $|f(\nabla\zeta_\lambda(x))| \leq 3|\nabla\zeta_\lambda(x)| \leq 3\kappa$. Therefore

$$|\det(\nabla\zeta_\lambda(x) - \nu_\beta, \nu_\gamma - \nu_\beta, \nu_\eta - \nu_\beta)| \geq |\det(\nu_\beta, \nu_\gamma, \nu_\eta)| - |f(\nabla\zeta_\lambda(x))| \geq \vartheta^2 - 3\kappa.$$

We deduce that

$$|\det(\nu_\alpha - \nu_\beta, \nu_\gamma - \nu_\beta, \nu_\eta - \nu_\beta)| = \frac{|\det(\nabla\zeta_\lambda(x) - \nu_\beta, \nu_\gamma - \nu_\beta, \nu_\eta - \nu_\beta)|}{\sigma_\alpha} \geq 4(\vartheta^2 - 3\kappa).$$

On the other hand, note that there exist $x_\alpha, x_\beta, x_\gamma, x_\eta$ in $B(x, \lambda)$ such that

$$\zeta(x_\alpha) = \zeta_\alpha(x_\alpha), \zeta(x_\beta) = \zeta_\beta(x_\beta), \zeta(x_\gamma) = \zeta_\gamma(x_\gamma), \zeta(x_\eta) = \zeta_\eta(x_\eta).$$

Since

$$B = \begin{pmatrix} \zeta_\alpha(x) - \zeta_\alpha(x_\alpha) + \zeta(x_\alpha) - \zeta(x_\beta) + \zeta_\beta(x_\beta) - \zeta_\beta(x) \\ \zeta_\gamma(x) - \zeta_\gamma(x_\gamma) + \zeta(x_\gamma) - \zeta(x_\beta) + \zeta_\beta(x_\beta) - \zeta_\beta(x) \\ \zeta_\eta(x) - \zeta_\eta(x_\eta) + \zeta(x_\eta) - \zeta(x_\beta) + \zeta_\beta(x_\beta) - \zeta_\beta(x) \end{pmatrix},$$

we deduce that $|B| \leq 3\lambda$. Hence the linear system of equations $A\tilde{y} = B$ admits a unique solution which satisfies

$$|\tilde{y}| = |y - x| = |A^{-1}B| \leq \frac{C\lambda}{\vartheta^2 - 3\kappa}.$$

Summarizing, if $x \in C_\kappa$ with $0 < \kappa < \vartheta^2/3$ then there exist $\alpha, \beta, \gamma, \eta \in \Lambda$ with $\alpha \neq \beta \neq \gamma \neq \eta$ such that the unique solution $y \in \mathbb{R}^3$ to (A.7) lies in the ball $B(x, C\lambda/(\vartheta^2 - 3\kappa))$. We conclude that the set C_κ can be covered by \mathfrak{B}_κ , a collection of at most $\binom{|\Lambda|}{4} \leq (2k)^8$ balls of radius $C\lambda/(\vartheta^2 - 3\kappa)$.

Observing that

$$|D^2\zeta_\lambda(x)| \leq \frac{C}{\lambda^2}$$

for any $x \in \mathbb{R}^3$, and letting

$$T_\kappa = \zeta_\lambda(\cup_{B \in \mathfrak{B}_\kappa} B) \cup \zeta_\lambda(P_\lambda),$$

we conclude that

$$|T_\kappa| \leq C(2k)^8 \frac{\lambda}{\vartheta^2 - 3\kappa}$$

and that, for any $t \in \mathbb{R} \setminus T_\kappa$, $\{x \mid \zeta_\lambda(x) = t\}$ is a complete submanifold of \mathbb{R}^3 whose second fundamental form is bounded by

$$C \frac{\sup_{\mathbb{R}^3} |D^2\zeta_\lambda|}{\inf_{\mathbb{R}^3 \setminus ((\cup_{B \in \mathfrak{B}_\kappa} B) \cup P_\lambda)} |\nabla\zeta_\lambda|} \leq \frac{C}{\lambda^2\kappa}.$$

Recalling the relation between λ and ϑ (see (A.2)), the proposition follows. \square

APPENDIX B. SMOOTH APPROXIMATION OF THE FUNCTION ζ FOR THE DISTANCE THROUGH THE BOUNDARY – FIRST METHOD

In this section of the Appendix we prove Proposition 5.2. We will smoothly approximate the function ζ for $d_{\partial\Omega}$ by convolution, after displacing the points a_i as in Appendix A. The main points of the proof are:

- Since $\partial\Omega$ is assumed to be of class C^2 , if we reduce the analysis to a small neighborhood close to the boundary then the gradient of the distance to the boundary at every point of this neighborhood is given by the normal to the boundary at the unique projection to the boundary of this point.

- We will characterize the set where the distance to the boundary is equal to one or two of the functions $\zeta_{i,j}$'s, while the gradient vectors of these functions do not satisfy “good” angle conditions between each other in the sense described in the previous section. One can show that the image of this set (by our smooth approximation) has small measure. To prove this fact, we will present an argument based on the curvature of the boundary. The assumption that $\partial\Omega$ is of class C^2 gives an upper bound for the maximal principal curvature at each point of the boundary, which roughly speaking implies that the boundary “cannot wiggle too much”.
- We will adapt the last part of the proof of Proposition 5.1. Arguing in the same fashion, but using the inverse function theorem, we can show that the set where the distance to the boundary is equal to three of the functions $\zeta_{i,j}$'s can be covered by a quantitative number of small balls.

Proof. Let us begin by defining the function ζ_λ . Let $\mathcal{A}' = \{p'_1, \dots, p'_k, n'_1, \dots, n'_k\}$ be the collection of points given by Proposition A.1 with $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$. Observe that $D_{\mathcal{A}} \leq \text{diam}(\Omega)$. For any $0 < \vartheta \leq C_0(2k)^{-5}$, where $C_0 = \min(C_1, \vartheta_1)$, we have

$$\begin{aligned} L_{\partial\Omega}(\mathcal{A}') &\leq \sum_{i=1}^k d_{\partial\Omega}(p'_i, n'_i) \leq \sum_{i=1}^k d_{\partial\Omega}(p_i, n_i) + d_{\partial\Omega}(p_i, p'_i) + d_{\partial\Omega}(n_i, n'_i) \\ &\leq L_{\partial\Omega}(\mathcal{A}) + C \text{diam}(\Omega)(2k)^6 \vartheta. \end{aligned}$$

An analogous argument shows that $L_{\partial\Omega}(\mathcal{A}) \leq L_{\partial\Omega}(\mathcal{A}') + C \text{diam}(\Omega)(2k)^6 \vartheta$. Therefore

$$(B.1) \quad |L_{\partial\Omega}(\mathcal{A}) - L_{\partial\Omega}(\mathcal{A}')| \leq C(2k)^6 \vartheta,$$

where throughout the proof $C > 0$ denotes a constant depending only on $\partial\Omega$ that may change from line to line.

Remember that by Lemma 5.3 there exists a 1-Lipschitz function $\zeta^* : \cup_{i=1, \dots, k} \{p'_i, n'_i\} \rightarrow \mathbb{R}$ such that

$$L_{\partial\Omega}(\mathcal{A}') = \sum_{i=1}^k \zeta^*(p'_i) - \zeta^*(n'_i).$$

Define the function ζ for $d_{\partial\Omega}$ as in Definition 5.2, i.e. set

$$\zeta(x) := \max_{i \in \{1, \dots, k\}} (\zeta^*(p'_i) - d_i(x, \partial\Omega)),$$

where

$$d_i(x, \partial\Omega) := \min \left[\max \left(\max_{j \in \{1, \dots, 2k\}} d_{(i,j)}(x), d(p'_i, \partial\Omega) - d(x, \partial\Omega) \right), d(p'_i, \partial\Omega) + d(x, \partial\Omega) \right]$$

and

$$d_{(i,j)}(x) := \langle p'_i - x, \nu_{(i,j)} \rangle, \quad \nu_{(i,j)} = \begin{cases} \frac{p'_i - a'_j}{|p'_i - a'_j|} & \text{if } p'_i \neq a'_j \\ 0 & \text{if } p'_i = a'_j \end{cases}.$$

Lemma 5.4 yields that $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a 1-Lipschitz function such that

$$\sum_{i=1}^k \zeta(p'_i) - \zeta(n'_i) = \sum_{i=1}^k \zeta^*(p'_i) - \zeta^*(n'_i) = L_{\partial\Omega}(\mathcal{A}').$$

Next, we regularize the function ζ . Let $\varphi \in C_c^\infty(B(0, 1), \mathbb{R}_+)$ be a mollifier such that $\int_{\mathbb{R}^3} \varphi(x) dx = 1$. Letting

$$(B.2) \quad \lambda := \vartheta^{1/\rho} \quad \text{for } \rho \in (0, 1/4),$$

we define

$$\zeta_\lambda(\cdot) := \varphi_\lambda * \zeta(\cdot) = \int_{\mathbb{R}^3} \varphi_\lambda(\cdot - y) \zeta(y) dy \quad \text{with } \varphi_\lambda(\cdot) = \frac{1}{\lambda} \varphi\left(\frac{\cdot}{\lambda}\right).$$

Argument for the first statement. Observe that $\|\zeta - \zeta_\lambda\|_{L^\infty(\mathbb{R}^3)} \leq \lambda$ from which we deduce that

$$(B.3) \quad \left| L_{\partial\Omega}(\mathcal{A}') - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq 2k\lambda.$$

By combining (B.1) with (B.3), we obtain

$$(B.4) \quad \left| L_{\partial\Omega}(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq C(2k)^6 \vartheta + 2k\lambda.$$

On the other hand, note that

$$\left| \sum_{i=1}^k \zeta(p_i) - \zeta(n_i) - \sum_{i=1}^k \zeta(p'_i) - \zeta(n'_i) \right| \leq \sum_{i=1}^k |p_i - n_i| + |p'_i - n'_i| \leq C(2k)^6 \vartheta.$$

By combining the previous estimate with (B.3), we get

$$(B.5) \quad \left| \sum_{i=1}^k \zeta(p_i) - \zeta(n_i) - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq C(2k)^6 \vartheta + 2k\lambda.$$

Then from (B.4) and (B.5), we deduce that

$$\left| L_{\partial\Omega}(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p_i) - \zeta_\lambda(n_i) \right| \leq C(2k)^6 \lambda^\rho.$$

Argument for the second statement. Note that, for any $x \in \mathbb{R}^3$,

$$(B.6) \quad \nabla \zeta_\lambda(x) = \int_{B(x, \lambda)} \varphi_\lambda(x - y) \nabla \zeta(y) dy.$$

Let us observe that, since $\partial\Omega$ is assumed to be of class C^2 , there exists a fixed number $\theta_0 = \theta_0(\partial\Omega)$ such that if $y \in \Omega$ satisfies $d(y, \partial\Omega) < \theta_0$ then

$$\nabla d(y, \partial\Omega) = \nu(z_y),$$

where throughout the proof $\nu(z_y)$ denotes the outer unit normal vector to $\partial\Omega$ at $z_y := \text{proj}_{\partial\Omega}(y)$.

We define

$$\Lambda := \{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq 2k, i \neq j\} \quad \text{and} \quad c := \max_{i \in \{1, \dots, k\}} (\zeta^*(p'_i) - d(p'_i, \partial\Omega)).$$

Then, letting

$$\begin{aligned} \zeta_{(i,j)}(\cdot) &:= \zeta^*(p'_i) - d_{(i,j)}(\cdot) \quad \text{for } (i, j) \in \Lambda \\ \zeta_+(\cdot) &:= c + d(\cdot, \partial\Omega) \\ \zeta_-(\cdot) &:= c - d(\cdot, \partial\Omega), \end{aligned}$$

observe that, for almost every $y \in \Omega$ such that $d(y, \partial\Omega) < \theta_0$,

$$\nabla \zeta(y) = \begin{cases} \nu_{(i,j)} & \text{if } \zeta(y) = \zeta_{(i,j)}(y) \quad \text{for some } (i, j) \in \Lambda \\ \nu(z_y) & \text{if } \zeta(y) = \zeta_+(y) \\ -\nu(z_y) & \text{if } \zeta(y) = \zeta_-(y). \end{cases}$$

In particular, $|\nabla \zeta(y)| = 1$ for almost every y as above. Thus

$$|\nabla \zeta_\lambda(x)| \leq \int_{B(x, \lambda)} \varphi_\lambda(x - y) |\nabla \zeta(y)| dy \leq \int_{B(x, \lambda)} \varphi_\lambda(x - y) dy = 1$$

for any

$$(B.7) \quad x \in \Omega_\lambda = \{x \in \Omega \mid 2\lambda^\rho < d(x, \partial\Omega) < \theta_0 - 2\lambda^\rho\}.$$

Argument for the third statement. Observe that

$$\zeta(x) = c \quad \text{for any } x \in \partial\Omega.$$

Thus

$$|\zeta_\lambda(\{x \in \Omega \mid d(x, \partial\Omega) \leq 2\lambda^\rho\})| \leq C\lambda^\rho.$$

Argument for the fourth statement. Let us first prove that there exists a set $P_\lambda \subset \mathbb{R}^3$ such that $|\zeta_\lambda(P_\lambda)| \leq C(2k)^4 \lambda^{3\rho/4}$ and that, for any $0 < \kappa < \vartheta^2/3$,

$$C_\kappa := \{x \in \Omega_\lambda \mid |\nabla \zeta_\lambda(x)| < \kappa\} \setminus P_\lambda$$

can be covered by \mathfrak{B}_κ , a collection of at most $C((2k)^8 + \theta_0(2k)^6(\vartheta^2 - 3\kappa)^{-3})$ balls of radius $C\lambda/(\vartheta^2 - 3\kappa)$. We will then conclude the fourth statement from this.

We follow the same strategy as in the proof of the third statement of Proposition 5.1. We start by defining the set P_λ . First, we let

$$P_\lambda^\Lambda := \{z \in \mathbb{R}^3 \mid d(z, P) \leq 2\lambda\},$$

where

$$P := \cup_{1 \leq i < j \leq k} P_{i,j}, \quad P_{i,j} := \{y \in \mathbb{R}^3 \mid \zeta_{(i,j)}(y) = \zeta_{(j,i)}(y)\}.$$

Arguing as in the proof of Proposition 5.1, we deduce that $|\zeta_\lambda(P_\lambda^\Lambda)| \leq C\lambda k^2$.

Let us also define

$$P_{\lambda,\pm}^{\text{dipole}} := \{x \in \Omega_\lambda \setminus P_\lambda^\Lambda \mid \text{there exist } y_1, y_2 \in B(x, 2\lambda), \alpha \in \Lambda \text{ such that} \\ \zeta_\lambda(y_1) = \zeta_\alpha, \zeta_\lambda(y_2) = \zeta_\pm, \text{ and } |\nu_\alpha \times \pm\nu(z_{y_2})| < \vartheta\}$$

and

$$P_{\lambda,\pm}^{\text{tripole}} := \{x \in \Omega_\lambda \setminus P_\lambda^\Lambda \mid \text{there exist } y_1, y_2, y_3 \in B(x, 2\lambda), \alpha, \beta \in \Lambda, \alpha \neq \beta, \text{ such that} \\ \zeta_\lambda(y_1) = \zeta_\alpha, \zeta_\lambda(y_2) = \zeta_\beta, \zeta_\lambda(y_3) = \zeta_\pm, \text{ and } |\det(\nu_\alpha, \nu_\beta, \pm\nu(z_{y_3}))| < \vartheta^2\}.$$

Finally, we let

$$P_\lambda := P_\lambda^\Lambda \cup P_{\lambda,+}^{\text{dipole}} \cup P_{\lambda,-}^{\text{dipole}} \cup P_{\lambda,+}^{\text{tripole}} \cup P_{\lambda,-}^{\text{tripole}}.$$

We claim that

$$|\zeta_\lambda(P_{\lambda,+}^{\text{dipole}} \cup P_{\lambda,-}^{\text{dipole}} \cup P_{\lambda,+}^{\text{tripole}} \cup P_{\lambda,-}^{\text{tripole}})| \leq C(2k)^4 \lambda^{3\rho/4},$$

which in turn implies that

$$(B.8) \quad |P_\lambda| \leq C(2k)^4 \vartheta^{3/4} = C(2k)^4 \lambda^{3\rho/4}.$$

Our argument is based on the curvature of $\partial\Omega$. Given a point $z \in \partial\Omega$, we denote by $k_{\min}(z)$ the minimal principal curvature of $\partial\Omega$ at z . We also denote by $r_{\min}(z)$ (resp. $r_{\max}(z)$) the minimal (resp. maximal) principal radii of curvature at z . Let us observe that since Ω is of class C^2 , for any point $z \in \partial\Omega$, $r_{\min}(z) \geq C > 0$.

We next study the sets $P_{\lambda,\pm}^{\text{dipole}}$. Given $x \in P_{\lambda,\pm}^{\text{dipole}}$, let us first assume that $|k_{\min}(z_{y_2})| > \vartheta^{1/4}$. By definition of r_{\max} and by recalling that $|v_1 \times v_2| = |v_1||v_2| \sin \theta$ with θ being the angle formed by v_1 and v_2 , we deduce that there exists a constant $C_1(\partial\Omega)$ such that, for any y satisfying

$$|y_2 - y| = C_1 r_{\max}(z_{y_2}) \vartheta,$$

we have

$$|\nu_\alpha \times \pm\nu(z_y)| \geq \vartheta.$$

Moreover, since $r_{\min}(z_{y_2}) \geq C > 0$, we deduce that there exists $C_2(\partial\Omega)$ such that, for any

$$C_1 r_{\max}(z_{y_2}) \vartheta \leq |y_2 - y| \leq C_2,$$

we have

$$|\nu_\alpha \times \pm\nu(z_y)| \geq \vartheta.$$

Noting that $r_{\max}(z_{y_2}) < 1/\vartheta^{1/4}$, we deduce that $\{x \in P_{\lambda,\pm}^{\text{dipole}} \mid |k_{\min}(z_{y_2})| > \vartheta^{1/4}\}$ can be covered by $\text{vol}(\Omega_\lambda) C_2^{-3} |\Lambda|$ balls of radius $C_1 \vartheta^{3/4}$.

Let us now assume that $|k_{\min}(z_{y_2})| \leq \vartheta^{1/4}$. In this case, the boundary “looks” flat around z_{y_2} in the principal direction of minimal curvature. This, combined with the fact that the minimal principal radii of curvature is bounded below, immediately implies that the number of connected components of $\{x \in P_{\lambda, \pm}^{\text{dipole}} \mid |k_{\min}(z_{y_2})| \leq \vartheta^{1/4}\}$ is bounded above by a constant that only depends on $\partial\Omega$. Then, it is not hard to see that

$$|\zeta_\lambda(\{x \in P_{\lambda, \pm}^{\text{dipole}} \mid |k_{\min}(z_{y_2})| \leq \vartheta^{1/4}\})| \leq C\vartheta.$$

Thus

$$|\zeta_\lambda(P_{\lambda, \pm}^{\text{dipole}})| \leq C(2k)^2\vartheta^{3/4}.$$

Let us now study the sets $P_{\lambda, \pm}^{\text{tripole}}$. By recalling that $|\det(v_1, v_2, v_3)| = |(v_1 \times v_2) \cdot v_3| = |v_1 \times v_2||v_3| |\cos \theta|$ with θ being the angle formed by $v_1 \times v_2$ and v_3 , arguing similarly as above one can check that

$$\{x \in P_{\lambda, \pm}^{\text{tripole}} \mid |k_{\min}(z_{y_3})| > \vartheta^{1/4}\}$$

can be covered by $\text{vol}(\Omega_\lambda)\tilde{C}_2^{-3}\binom{|\Lambda|}{2}$ balls of radius $\tilde{C}_1\vartheta^{3/4}$. Moreover, arguing as above we deduce that

$$|\zeta_\lambda(\{x \in P_{\lambda, \pm}^{\text{tripole}} \mid |k_{\min}(z_{y_3})| \leq \vartheta^{1/4}\})| \leq C\vartheta.$$

Thus

$$|\zeta_\lambda(P_{\lambda, \pm}^{\text{tripole}})| \leq C(2k)^4\vartheta^{3/4},$$

which concludes the proof of the claim.

Let us now give a characterization of the gradient of ζ_λ for points $x \in \Omega_\lambda$. From (B.6), we deduce that

$$(B.9) \quad \nabla\zeta_\lambda(x) = \sum_{\alpha \in \Lambda} \sigma_\alpha \nu_\alpha + \int_{B(x, \lambda)} \varphi_\lambda(x-y)\nu(z_y)\mathbf{1}_{\zeta(y)=\zeta_+(y)}dy \\ + \int_{B(x, \lambda)} \varphi_\lambda(x-y)(-\nu(z_y))\mathbf{1}_{\zeta(y)=\zeta_-(y)}dy,$$

where

$$\sigma_\alpha = \int_{B(x, \lambda)} \varphi_\lambda(x-y)\mathbf{1}_{\zeta(y)=\zeta_\alpha(y)}dy \quad \text{for } \alpha \in \Lambda.$$

Let us observe that if there exists a point $x \in \Omega_\lambda$ such that $\zeta_\lambda(x) = \pm d(x, \partial\Omega)$ then for any $y \in B(x, \lambda)$, $\zeta_\lambda(y) \neq \mp d(y, \partial\Omega)$. This implies that if the second term in the right-hand side of (B.9) is different from zero then the third term vanishes, and viceversa. For this reason, without loss of generality we assume in what follows that the third term in the right-hand side of (B.9) vanishes.

Let us observe that, if there exists $y_x \in B(x, \lambda)$ such that $\zeta_\lambda(y_x) = d(y_x, \partial\Omega)$ then

$$\begin{aligned} \int_{B(x, \lambda)} \varphi_\lambda(x-y) \nu(z_y) \mathbf{1}_{\zeta(y)=\zeta_+(y)} dy &= \nu(z_{y_x}) \int_{B(x, \lambda)} \varphi_\lambda(x-y) \mathbf{1}_{\zeta(y)=\zeta_+(y)} dy \\ &\quad + \int_{B(x, \lambda)} \varphi_\lambda(x-y) (\nu(z_y) - \nu(z_{y_x})) \mathbf{1}_{\zeta(y)=\zeta_+(y)} dy, \end{aligned}$$

but $|\nu(z_y) - \nu(z_{y_x})| \leq C\lambda$, and therefore

$$\int_{B(x, \lambda)} \varphi_\lambda(x-y) \nu(z_y) \mathbf{1}_{\zeta(y)=\zeta_+(y)} dy = \sigma_+(1 + O(\lambda))\nu(z_{y_x}),$$

where

$$\sigma_+ := \int_{B(x, \lambda)} \varphi_\lambda(x-y) \mathbf{1}_{\zeta(y)=\zeta_+(y)} dy.$$

Thus, (B.9) can be written as

$$\nabla \zeta_\lambda(x) = \sum_{\alpha \in \Lambda} \sigma_\alpha \nu_\alpha + \sigma_+(1 + O(\lambda))\nu(z_{y_x}),$$

where y_x is arbitrarily chosen among the points in $y \in B(x, \lambda)$ such that $\zeta_\lambda(y) = d(y, \partial\Omega)$.

Now consider a number $0 < \kappa < \vartheta^2/3$ and a point $x \in C_\kappa$. We write

$$\nabla \zeta_\lambda(x) = \sum_{\alpha \in \Lambda} \sigma_\alpha \nu_\alpha + \sigma_+(1 + O(\lambda))\nu(z_{y_x})$$

and observe that since $x \notin P_\lambda^\Lambda$, if $\sigma_{(i,j)} > 0$ for some $(i, j) \in \Lambda$ then $\sigma_{(j,i)} = 0$.

We will consider two cases. First, if $\sigma_\alpha = 0$ then, arguing exactly as in the proof of Proposition 5.1, we conclude that there exist $\alpha, \beta, \gamma, \eta \in \Lambda$ with $\alpha \neq \beta \neq \gamma \neq \eta$ such that

$$\nabla \zeta_\lambda(x) = \sigma_\alpha \nu_\alpha + \sigma_\beta \nu_\beta + \sigma_\gamma \nu_\gamma + \sigma_\eta \nu_\eta$$

for some $\sigma_\alpha, \sigma_\beta, \sigma_\gamma, \sigma_\eta \in (0, 1)$ with $\sigma_\alpha + \sigma_\beta + \sigma_\gamma + \sigma_\eta = 1$, and

$$|\det(\nu_\alpha - \nu_\beta, \nu_\gamma - \nu_\beta, \nu_\eta - \nu_\beta)| \geq 3(\vartheta^2 - 3\kappa).$$

Therefore the unique solution $y \in \mathbb{R}^3$ to (A.7) lies in the ball $B(x, C\lambda/(\vartheta^2 - 3\kappa))$.

Second, if $\sigma_\alpha > 0$ then, using the fact that $x \notin P_\lambda \setminus P_\lambda^\Lambda$ and arguing as in the proof of Proposition 5.1, we conclude that

$$\nabla \zeta_\lambda(x) = \sigma_+(1 + O(\lambda))\nu(z_{y_x}) + \sigma_\alpha \nu_\alpha + \sigma_\beta \nu_\beta + \sigma_\gamma \nu_\gamma,$$

for some $\sigma_+, \sigma_\alpha, \sigma_\beta, \sigma_\gamma \in (0, 1)$ with $\sigma_+ + \sigma_\alpha + \sigma_\beta + \sigma_\gamma = 1$, and

$$(B.10) \quad |\det(\nu(z_{y_x}) - \nu_\beta, \nu_\alpha - \nu_\beta, \nu_\gamma - \nu_\beta)| \geq 3(\vartheta^2 - 3\kappa).$$

Let us consider the function $\Phi : \Omega_\lambda \rightarrow \mathbb{R}^3$ defined via

$$\Phi(\cdot) = (\zeta_+ - \zeta_\beta, \zeta_\alpha - \zeta_\beta, \zeta_\gamma - \zeta_\beta)(\cdot).$$

Observe that

$$D\Phi(x) = (\nu(z_{yx}) - \nu_\beta, \nu_\alpha - \nu_\beta, \nu_\gamma - \nu_\beta),$$

whose determinant satisfies (B.10). Moreover, by noting that

$$|D\Phi(x) - D\Phi(y)| = |(\nu(z_{yx}) - \nu(z_y), 0, 0)|,$$

we easily deduce that

$$(B.11) \quad |\det(\nu(z_y) - \nu_\beta, \nu_\alpha - \nu_\beta, \nu_\gamma - \nu_\beta)| \geq (\vartheta^2 - 3\kappa)$$

for any $y \in B(x, C(\vartheta^2 - 3\kappa))$. Let us observe that our definition of Ω_λ (see (B.7)) guaranties that $B(x, C(\vartheta^2 - 3\kappa)) \subset B(x, C\lambda^{2\rho}) \subset \{x \in \Omega \mid d(x, \partial\Omega) < \theta_0\}$ for any sufficiently small λ . We also observe that, since there exist $x_+, x_\alpha, x_\beta, x_\gamma \in B(x, \lambda)$ such that

$$\zeta(x_+) = \zeta_+(x_+), \zeta(x_\alpha) = \zeta_\alpha(x_\alpha), \zeta(x_\beta) = \zeta_\beta(x_\beta), \zeta(x_\gamma) = \zeta_\gamma(x_\gamma),$$

we have

$$(B.12) \quad |\Phi(x)| \leq C\lambda.$$

From a “quantitative version” of the inverse function theorem (see [Lan93, Chapter XIV, Lemma 1.3]), we conclude that Φ is invertible in $B(x, C(\vartheta^2 - 3\kappa))$, and since $\lambda \ll \vartheta^4$ (recall that $\rho < 1/4$), $0 \in \Phi(B(x, C(\vartheta^2 - 3\kappa)))$. In particular, there exists a unique $y \in B(x, C(\vartheta^2 - 3\kappa))$ such that $\Phi(y) = 0$. Moreover

$$|x - y| = |\Phi^{-1}(\Phi(x)) - \Phi^{-1}(0)| \leq |D\Phi^{-1}(z)| |\Phi(x) - 0|$$

for some $z \in \Phi(B(x, C(\vartheta^2 - 3\kappa)))$. This, combined with (B.11) and (B.12), gives

$$|x - y| \leq \frac{C\lambda}{\vartheta^2 - 3\kappa}.$$

This means that the unique solution to $\Phi(y) = 0$ in the ball $B(x, C(\vartheta^2 - 3\kappa))$ lies in the much smaller ball $B(x, \frac{C\lambda}{\vartheta^2 - 3\kappa})$.

Since Ω_λ can be covered by $C\theta_0(\vartheta^2 - 3\kappa)^{-3}$ balls of radius $\vartheta^2 - 3\kappa$, we deduce that C_κ can be covered by \mathfrak{B}_κ , a collection of at most

$$\binom{|\Lambda|}{4} + C\theta_0(\vartheta^2 - 3\kappa)^{-3} \binom{|\Lambda|}{3}$$

balls of radius $C\lambda/(\vartheta^2 - 3\kappa)$.

Observing that

$$|D^2\zeta_\lambda(x)| \leq \frac{C}{\lambda^2}$$

for any $x \in \Omega_\lambda$, and letting

$$T_\kappa := \zeta_\lambda(\cup_{B \in \mathfrak{B}_\kappa} B) \cup \zeta_\lambda(P_\lambda),$$

we conclude that (recall (B.8))

$$|T_\kappa| \leq C \left((2k)^8 \frac{\lambda}{\vartheta^2 - 3\kappa} + (2k)^6 \frac{\lambda}{(\vartheta^2 - 3\kappa)^4} + (2k)^4 \lambda^{3\rho/4} \right)$$

and that, for any $t \in \zeta_\lambda(\Omega_\lambda) \setminus T_\kappa$, $\{x \mid \zeta_\lambda(x) = t\}$ is a complete submanifold of \mathbb{R}^3 whose second fundamental form is bounded by

$$C \frac{\sup_{\Omega_\lambda} |D^2 \zeta_\lambda|}{\inf_{\Omega_\lambda \setminus ((\cup_{B \in \mathfrak{B}_\kappa} B) \cup P_\lambda)} |\nabla \zeta_\lambda|} \leq \frac{C}{\lambda^2 \kappa}.$$

Recalling the relation between λ and ϑ (see (B.2)), the proposition follows. \square

APPENDIX C. SMOOTH APPROXIMATION OF THE FUNCTION ζ FOR THE DISTANCE THROUGH THE BOUNDARY – SECOND METHOD

In this section of the Appendix we prove Proposition 5.3. By assuming that the Gauss curvature of the boundary of Ω is strictly positive, we will provide a convex polyhedral approximation of $\partial\Omega$, very close in Hausdorff distance. We will then smoothly approximate the function ζ for the distance through the polyhedral approximation of $\partial\Omega$ by convolution, after performing a suitable displacement of the points of the collection. The main points of the proof are:

- The commodity of replacing the boundary of the domain by a convex polyhedron is that, where well-defined, the gradient of the function distance to the polyhedron is equal to the normal to one of its faces.
- The strategy of proof is very similar to the one followed to prove Proposition 5.1. But in this case to study the set of points whose gradient is small, we need to ensure that the normals to the faces of the convex polyhedral approximation of $\partial\Omega$ and the vectors $\nu_{(i,j)}$ satisfy “good” angle conditions between each other. To accomplish this we will carefully choose the approximating convex polyhedron, and then perform a displacement of the points of the configuration \mathcal{A} .

C.1. Polyhedral approximation of the boundary. We denote by $\mathcal{X} = \{x_1, \dots, x_n\}$ a collection of points belonging to $\partial\Omega$ such that

$$(C.1) \quad \frac{3}{2}\tau \leq \min_{1 \leq i \neq j \leq n} \mathfrak{d}(x_i, x_j) \quad \text{and} \quad \forall z \in \partial\Omega, \mathfrak{d}(z, x_i) \leq \frac{5}{2}\tau \text{ for some } x_i \in \mathcal{X},$$

where from now on \mathfrak{d} denotes the geodesic distance on $\partial\Omega$ and $\tau \in (0, 1)$ is a given number. For any $x_i \in \mathcal{X}$ let us denote by $\nu(x_i)$ the outer unit normal to $\partial\Omega$. Define

$$\Omega_{\mathcal{X}} := \cap_{1 \leq i \leq n} \{z \mid \langle z - x_i, \nu(x_i) \rangle < 0\}.$$

It is easy to see that $\partial\Omega_{\mathcal{X}}$ is a polyhedral approximation of $\partial\Omega$ which in addition is convex if Ω is convex.

In the next lemma we show that the points of the collection \mathcal{X} can be displaced in order to make the normals $\nu(x_i)$'s satisfy “good” angle conditions between each other, when Ω is assumed to have strictly positive Gauss curvature.

Lemma C.1. *Let Ω be a C^2 domain and assume that $\partial\Omega$ has strictly positive Gauss curvature. Let $\mathcal{X} = \{x_1, \dots, x_n\}$ be a collection of points belonging to $\partial\Omega$ satisfying (C.1) for a number $\tau \in (0, 1)$. Then there exist constants $\tau_0, C_0, C > 0$ depending only on $\partial\Omega$, such that for any $0 < \tau < \tau_0$ there exists a collection $\mathcal{X}' = \{y_1, \dots, y_n\} \subset \partial\Omega$ such that, for any $0 < \vartheta \leq C_0\tau^5$, the following hold:*

- (1) $\tau \leq \min_{1 \leq i \neq j \leq n} \mathfrak{d}(y_i, y_j)$ and, for any $z \in \partial\Omega$, $\mathfrak{d}(z, y_i) \leq 3\tau$ for some $y_i \in \mathcal{X}'$.
- (2) Letting

$$\Omega_{\mathcal{X}'} := \bigcap_{1 \leq i \leq n} \{z \mid \langle z - y_i, \nu(y_i) \rangle < 0\},$$

where $\nu(y_i)$ is the outer unit normal to $\partial\Omega$ at y_i , we have

$$|d(z, \partial\Omega_{\mathcal{X}'}) - d(z, \partial\Omega)| \leq C\tau^2 \quad \text{for any } z \in \mathbb{R}^3.$$

- (3) For any $i, j, k \in \{1, \dots, n\}$ with $i \neq j \neq k$, we have

$$|\nu(y_i) \times \nu(y_j)| \geq \vartheta \quad \text{and} \quad |\det(\nu(y_i), \nu(y_j), \nu(y_k))| \geq \vartheta^2.$$

Proof. Since we assume that $\partial\Omega$ has strictly positive Gauss curvature, we deduce that there exists a constant $\kappa_0(\partial\Omega) > 0$ such that for any point $x \in \partial\Omega$ the minimal principal curvature of $\partial\Omega$ at x is bounded below by κ_0 . In addition, since $\partial\Omega$ is assumed to be of class C^2 , we deduce that for any point $x \in \partial\Omega$ the maximal principal curvature of $\partial\Omega$ at x is bounded above by a certain constant $K_0(\partial\Omega) > 0$. From these two facts, we deduce that there exists constants $\vartheta_0, C_1 > 0$ depending only on $\partial\Omega$, such that for any $x \in \partial\Omega$ and for any $0 < \vartheta < \vartheta_0$, if $v \in \mathbb{R}^3$ with $|v| = 1$ satisfies

$$\theta(v, \nu(x)) < \vartheta,$$

where $\theta(v, \nu(x))$ is the angle formed by v and $\nu(x)$, then for any $y \in \partial\Omega$ satisfying

$$C_1\kappa_0^{-1}\vartheta \leq \mathfrak{d}(x, y) \leq C_1^{-1}K_0^{-1},$$

we have

$$\theta(v, \nu(y)) \geq \vartheta.$$

Since $|v_1 \times v_2| = |v_1||v_2||\sin \theta|$ with θ being the angle formed by v_1 and v_2 , we easily deduce that, up to an adjustment of C_1 , for any $x \in \partial\Omega$ and for any $0 < \vartheta \leq \vartheta_0$, if $v \in \mathbb{R}^3$ with $|v| = 1$ is such that

$$|v \times \nu(x)| < \vartheta,$$

then, letting

$$D_{x, \vartheta} = \{y \in \partial\Omega \mid C_1\kappa_0^{-1}\vartheta \leq \mathfrak{d}(x, y) \leq C_1^{-1}K_0^{-1}\},$$

we have

$$|v \times \nu(y)| \geq \vartheta$$

for any $y \in D_{x,\vartheta}$.

On the other hand, by recalling that $|\det(v_1, v_2, v_3)| = |v_1 \times v_2| |v_3| |\cos \theta|$ with θ being the angle formed by $v_1 \times v_2$ and v_3 , we deduce that, up a further adjustment of C_1 , for any $x \in \partial\Omega$ and for any $0 < \vartheta \leq \vartheta_0$, if $v, w \in \mathbb{R}^3$ with $|v| = |w| = 1$ are such that

$$|v \times w| \geq \vartheta \quad \text{and} \quad |\det(v, w, \nu(x))| < \vartheta^2,$$

then for any $y \in D_{x,\vartheta}$ we have

$$|\det(v, w, \nu(y))| \geq \vartheta^2.$$

Given a point $x \in \partial\Omega$ and a unit vector $v \in \mathbb{R}^3$, we define

$$D_{x,v,\vartheta} = \{y \in \partial\Omega \mid \mathfrak{d}(x, y) \leq K_0^{-1}, |v \times \nu(y)| < \vartheta\}.$$

By the previous arguments, we deduce that $D_{x,v,\vartheta} = \emptyset$ or

$$D_{x,v,\vartheta} \subseteq B_{\mathfrak{d}}(y_x, C_1^{-1}K_0^{-1}) \setminus D_{y_x,\vartheta}$$

for some $y_x \in \partial\Omega$, where $B_{\mathfrak{d}}(y_x, C_1^{-1}K_0^{-1})$ denotes a ball (with respect to the distance \mathfrak{d}) centered at y_x of radius $C_1^{-1}K_0^{-1}$. In particular, $|D_{x,v,\vartheta}| \leq C\vartheta^2$, where hereafter C denotes a constant depending only on $\partial\Omega$ that may change from line to line.

Let us also define, given a point $x \in \partial\Omega$ and unit vectors $v_1, v_2 \in \mathbb{R}^3$ with $|v_1 \times v_2| \geq \vartheta$, the set

$$T_{x,v_1,v_2,\vartheta} = \{y \in \partial\Omega \mid \mathfrak{d}(x, y) \leq K_0^{-1}, |\det(v_1, v_2, \nu(y))| < \vartheta^2\}.$$

Arguing as above, we deduce that $|T_{x,v_1,v_2,\vartheta}| \leq C\vartheta^2$.

We will now proceed by induction. Define $y_1 = x_1$ and let $0 < \vartheta \leq C_0\tau^{-5}$ with $C_0 = \min(1, \vartheta_0)$. Assume that we have defined a collection $\{y_1, \dots, y_l\} \subset \partial\Omega$ with $1 < l < n$, such that for any $i, j, k \in \{1, \dots, l\}$ with $i \neq j \neq k$, we have

$$|\nu(y_i) \times \nu(y_j)| \geq \vartheta \quad \text{and} \quad |\det(\nu(y_i), \nu(y_j), \nu(y_k))| \geq \vartheta^2$$

and $\mathfrak{d}(x_i, y_i) \leq d_i := C_1\kappa_0^{-1}((i-1)\vartheta + (i-1)^2\vartheta)$ for any $i \in \{1, \dots, l\}$.

We will next find a point $y_{l+1} \in \partial\Omega$ with $\mathfrak{d}(x_{l+1}, y_{l+1}) \leq d_{l+1}$, such that the collection $\{y_1, \dots, y_l, y_{l+1}\} \subset \partial\Omega$ satisfies

$$|\nu(y_i) \times \nu(y_j)| \geq \vartheta \quad \text{and} \quad |\det(\nu(y_i), \nu(y_j), \nu(y_k))| \geq \vartheta^2$$

for any $i, j, k \in \{1, \dots, l+1\}$ with $i \neq j \neq k$.

Let us define

$$X_{\text{bad}}^{l+1} := \left(\bigcup_{i \in \{1, \dots, l\}} D_{x_{l+1}, \nu(y_i), \vartheta} \right) \cup \left(\bigcup_{i, j \in \{1, \dots, l\}, i \neq j} T_{x_{l+1}, \nu(y_i), \nu(y_j), \vartheta} \right).$$

Note that, for any $y \in B_{\mathfrak{d}}(x_{l+1}, K_0^{-1}) \setminus X_{\text{bad}}^{l+1}$, the collection of points $\{y_1, \dots, y_l, y\}$ satisfies

$$|\nu(y_i) \times \nu(y_j)| \geq \vartheta \quad \text{and} \quad |\det(\nu(y_i), \nu(y_j), \nu(y_k))| \geq \vartheta^2$$

for any $i, j, k \in \{1, \dots, l+1\}$ with $i \neq j \neq k$. Moreover, by definition of $D_{x_{l+1}, \nu(y_i), \vartheta}$ and $T_{x_{l+1}, \nu(y_i), \nu(y_j), \vartheta}$, we deduce that we can find a point $y_{l+1} \in B_\vartheta(x_{l+1}, K_0^{-1}) \setminus X_{\text{bad}}^{l+1}$, which in addition satisfies

$$\mathfrak{d}(x_{l+1}, y_{l+1}) \leq C_1 \kappa_0^{-1} (l\vartheta + l^2\vartheta).$$

This concludes the induction step and the proof of the third assertion. Let us now observe that, for any $l \in \{1, \dots, n\}$, we have

$$d_l \leq Cn^2\vartheta.$$

By noting that $n \leq c\tau^{-2}$ for some universal constant $c > 0$, we get

$$d_l \leq C\tau^{-4}\vartheta \leq CC_0\tau.$$

Therefore, up to an adjustment of C_0 , we have that $d_l \leq \frac{1}{4}\tau$, which ensures that the first assertion holds. Finally, see [Gru93, Theorem 4] for a proof of the second statement. This concludes the proof of the lemma. \square

C.2. Displacement of the points. With the aid of Lemmas A.1, A.2, and C.1, we perform the displacement of the points of the collection \mathcal{A} .

Proposition C.1. *Let Ω be a C^2 domain and assume that $\partial\Omega$ has strictly positive Gauss curvature. Let $\mathcal{A} = \{a_1, \dots, a_m\} \subset \Omega$ be a collection of m not necessarily distinct points. Consider a collection $\mathcal{X} = \{x_1, \dots, x_n\} \subset \partial\Omega$ satisfying (C.1) and let $\mathcal{X}' = \{y_1, \dots, y_n\} \subset \partial\Omega$ be the collection of points given by Lemma C.1 for a number $\tau < \tau_0$, where τ_0 is the constant appearing in the lemma. Then there exist constants $C_0, C_1 > 0$ depending only on $\partial\Omega$ and a collection of points $\mathcal{A}' = \{b_1, \dots, b_m\} \subset \Omega$ such that, for any*

$$\vartheta < C_0 \min\{m^{-5}, m^{-3}\tau^2, m^{-1}\tau^4, \tau^5\},$$

the following hold:

- (1) $b_i \neq b_j$ for any $i \neq j$.
- (2) Define

$$\nu_{(i,j)} := \frac{b_i - b_j}{|b_i - b_j|} \quad \text{for } (i,j) \in \Lambda_m := \{(p,q) \mid 1 \leq p < q \leq m\}$$

and

$$\mathcal{V} := \{\nu_{(i,j)} \mid (i,j) \in \Lambda_m\} \cup \{\nu(y_i) \mid y_i \in \mathcal{X}'\}.$$

Then for any $u, v, w \in \mathcal{V}$ with $u \neq v \neq w$, we have

$$|u \times v| \geq \vartheta \quad \text{and} \quad |\det(u, v, w)| \geq \vartheta^2.$$

- (3) $|a_l - b_l| \leq C_1(l^5 + l^3\tau^{-2} + l\tau^{-4})\vartheta$ for any $l \in \{1, \dots, m\}$.

Proof. First, let us observe that Lemmas A.1 and A.2 hold as well if we replace B by Ω and D by $D_\Omega := \text{diam}(\Omega)$. Let $0 < \vartheta \leq C_0 \min\{m^{-5}, m^{-3}\tau^2, m^{-1}\tau^4, \tau^5\}$ with C_0 sufficiently so that we can apply Lemmas A.1, A.2, and C.1. In particular, for any $i, j, k \in \{1, \dots, n\}$ with $i \neq j \neq k$, we have

$$|\nu(y_i) \times \nu(y_j)| \geq \vartheta \quad \text{and} \quad |\det(\nu(y_i), \nu(y_j), \nu(y_k))| \geq \vartheta^2.$$

We proceed by induction. We let $b_1 = a_1$ and assume that we have defined a collection $\{b_1, \dots, b_l\} \subset \Omega$ with $1 \leq l < m$ such that:

- Letting

$$\mathcal{Y}_l = \{\nu_{(i,j)} \mid (i,j) \in \Lambda_l\} \cup \{\nu(y_i) \mid y_i \in \mathcal{X}'\},$$

for any $u, v, w \in \mathcal{Y}_l$ with $u \neq v \neq w$, we have

$$|u \times v| \geq \vartheta \quad \text{and} \quad |\det(u, v, w)| \geq \vartheta^2.$$

- $|a_i - b_i| \leq C_1((i-1)^5 + (i-1)^3\tau^{-2} + (i-1)\tau^{-4})\vartheta$ for any $i \in \{1, \dots, l\}$, for some constant C_1 depending only on $\partial\Omega$.

We will next find $b_{l+1} \in \Omega$ such that the collection $\{b_1, \dots, b_l, b_{l+1}\} \subset \Omega$ satisfies that:

- For any $u, v, w \in \mathcal{Y}_{l+1}$ with $u \neq v \neq w$, we have

$$|u \times v| \geq \vartheta \quad \text{and} \quad |\det(u, v, w)| \geq \vartheta^2.$$

- $|a_{l+1} - b_{l+1}| \leq C_1(l^5 + l^3\tau^{-2} + l\tau^{-4})\vartheta$.

First, for any $i \in \{1, \dots, l\}$ and $\alpha \in \Lambda_l$, let $C_{\nu_\alpha, b_i, D_\Omega, \vartheta}$ denote the cylinder defined in Lemma A.1 with $\nu = \nu_\alpha$, B replaced by Ω , D replaced by D_Ω , and $x = b_i$. In addition, for any $i \in \{1, \dots, l\}$ and $\alpha, \beta \in \Lambda_l$ with $\alpha \neq \beta$, let $P_{\nu_\alpha, \nu_\beta, b_i, D_\Omega, \vartheta}$ denote the set defined in Lemma A.2 with $\nu_1 = \nu_\alpha$, $\nu_2 = \nu_\beta$, B replaced by Ω , D replaced by D_Ω , and $x = b_i$. We define

$$X_{\text{bad},1}^l := \left(\bigcup_{i \in \{1, \dots, l\}} \bigcup_{\alpha \in \Lambda_l} C_{\nu_\alpha, b_i, D_\Omega, \vartheta} \right) \cup \left(\bigcup_{i \in \{1, \dots, l\}} \bigcup_{\alpha, \beta \in \Lambda_l, \alpha \neq \beta} P_{\nu_\alpha, \nu_\beta, b_i, D_\Omega, \vartheta} \right).$$

By Lemmas A.1 and A.2, we conclude that, for any $y \in \Omega \setminus X_{\text{bad},1}^l$, the collection of points $\{b_1, \dots, b_l, y\}$ is such that, for any $\alpha, \beta, \gamma \in \Lambda_{l+1}$ with $\alpha \neq \beta \neq \gamma$,

$$|\nu_\alpha \times \nu_\beta| \geq \vartheta \quad \text{and} \quad |\det(\nu_\alpha, \nu_\beta, \nu_\gamma)| \geq \vartheta^2.$$

Second, for any $i \in \{1, \dots, l\}$ and $j \in \{1, \dots, n\}$, let $C_{\nu(y_j), b_i, D_\Omega, \vartheta}$ denote the cylinder defined in Lemma A.1 with $\nu = \nu(y_j)$, B replaced by Ω , D replaced by D_Ω , and $x = b_i$. In addition, for any $i \in \{1, \dots, l\}$ and $j, k \in \{1, \dots, n\}$ with $j \neq k$, let

$P_{\nu(y_j), \nu(y_k), b_i, D_\Omega, \vartheta}$ denote the set defined in Lemma A.2 with $\nu_1 = \nu(y_j)$, $\nu_2 = \nu(y_k)$, B replaced by Ω , D replaced by D_Ω , and $x = b_i$. We define

$$X_{\text{bad},2}^l := \left(\bigcup_{i \in \{1, \dots, l\}} \bigcup_{j \in \{1, \dots, n\}} C_{\nu(y_j), b_i, D_\Omega, \vartheta} \right) \cup \left(\bigcup_{i \in \{1, \dots, l\}} \bigcup_{j, k \in \{1, \dots, n\}, j \neq k} P_{\nu(y_j), \nu(y_k), b_i, D_\Omega, \vartheta} \right).$$

By Lemmas A.1 and A.2, we conclude that, for any $y \in \Omega \setminus X_{\text{bad},2}^l$, the collection of points $\{b_1, \dots, b_l, y\}$ is such that, for any $\alpha \in \Lambda_{l+1}$ and $j, k \in \{1, \dots, n\}$ with $j \neq k$,

$$|\nu_\alpha \times \nu(y_j)| \geq \vartheta \quad \text{and} \quad |\det(\nu_\alpha, \nu(y_j), \nu(y_k))| \geq \vartheta^2.$$

Third, for any $i \in \{1, \dots, l\}$, $j \in \{1, \dots, n\}$, and $\alpha \in \Lambda_l$, let $P_{\nu_\alpha, \nu(y_j), b_i, D_\Omega, \vartheta}$ denote the set defined in Lemma A.2 with $\nu_1 = \nu_\alpha$, $\nu_2 = \nu(y_j)$, B replaced by Ω , D replaced by D_Ω , and $x = b_i$. We define

$$X_{\text{bad},3}^l := \bigcup_{i \in \{1, \dots, l\}} \bigcup_{j \in \{1, \dots, n\}} \bigcup_{\alpha \in \Lambda_l} P_{\nu_\alpha, \nu(y_j), b_i, D_\Omega, \vartheta}.$$

By Lemmas A.1 and A.2, we conclude that, for any $y \in \Omega \setminus X_{\text{bad},3}^l$, the collection of points $\{b_1, \dots, b_l, y\}$ is such that, for any $\alpha, \beta \in \Lambda_{l+1}$ with $\alpha \neq \beta$ and $j \in \{1, \dots, n\}$,

$$|\det(\nu_\alpha, \nu_\beta, \nu(y_j))| \geq \vartheta^2.$$

Finally, we let $X_{\text{bad}}^l = X_{\text{bad},1}^l \cup X_{\text{bad},2}^l \cup X_{\text{bad},3}^l$ and observe that, for any $y \in \Omega \setminus X_{\text{bad}}^l$, the collection of points $\{b_1, \dots, b_l, y\}$ is such that, for any $u, v, w \in \mathcal{V}_{l+1}$ with $u \neq v \neq w$, we have

$$|u \times v| \geq \vartheta \quad \text{and} \quad |\det(u, v, w)| \geq \vartheta^2.$$

Let us note that

$$|X_{\text{bad}}^l| \leq C(l|\Lambda_l| + l|\Lambda_l|^2 + ln + ln^2 + ln|\Lambda_l|)\vartheta \leq C(l^5 + l^3n + ln^2)\vartheta,$$

where $C > 0$ is a constant that depends only on $\partial\Omega$. By recalling that $n \leq c\tau^{-2}$ for some universal constant $c > 0$ and arguing as in the proof of Proposition A.1, we deduce that, up to an adjustment of the constant C_0 , we have

$$|\Omega \setminus X_{\text{bad}}^l| \leq \frac{1}{10}|\Omega|$$

for any $0 < \vartheta \leq C_0 \min\{m^{-5}, m^{-3}\tau^2, m^{-1}\tau^4, \tau^5\}$. In addition, since $a_{l+1} \in \Omega$, we deduce that there exists a point $b_{l+1} \in \Omega \setminus X_{\text{bad}}^l$ such that

$$|a_{l+1} - b_{l+1}| \leq C_1(l^5 + l^3\tau^{-2} + l\tau^{-4})\vartheta,$$

where C_1 is a constant that depends only on $\partial\Omega$. We conclude that the collection $\{b_1, \dots, b_l, b_{l+1}\}$ satisfies the desired properties. This ends the proof of the induction step and the proposition is thus proved. \square

C.3. Proof of Proposition 5.3.

Proof. Let us begin by defining the function ζ_λ . Let $\mathcal{X} = \{x_1, \dots, x_n\} \subset \partial\Omega$ be a collection of points satisfying (C.1) for a number $\tau \in (0, 1)$. Apply Lemma C.1 to obtain a collection $\mathcal{X}' = \{y_1, \dots, y_n\}$ for $0 < \tau < \tau_0$, where τ_0 is the constant appearing in the statement of the lemma. Then apply Proposition C.1 with the collection of points $\mathcal{A} \subset \Omega$ to obtain a collection $\mathcal{A}' = \{p'_1, \dots, p'_k, n'_1, \dots, n'_k\} \subset \Omega$. For any

$$\vartheta < C_0 \min\{(2k)^{-5}, (2k)^{-5}\tau^2, (2k)^{-1}\tau^4, \tau^5\},$$

where $C_0 = C_0(\partial\Omega)$ is the constant appearing in the statement of the proposition, we have

$$\begin{aligned} L_{\partial\Omega}(\mathcal{A}') &\leq \sum_{i=1}^k d_{\partial\Omega}(p'_i, n'_i) \leq \sum_{i=1}^k d_{\partial\Omega}(p_i, n_i) + d_{\partial\Omega}(p_i, p'_i) + d_{\partial\Omega}(n_i, n'_i) \\ &\leq L_{\partial\Omega}(\mathcal{A}) + C(2k) \left((2k)^5 + (2k)^3\tau^{-2} + (2k)\tau^{-4} \right) \vartheta, \end{aligned}$$

where throughout the proof C denotes a constant depending only on $\partial\Omega$, that may change from line to line. An analogous argument shows that

$$L_{\partial\Omega}(\mathcal{A}) \leq L_{\partial\Omega}(\mathcal{A}') + C(2k) \left((2k)^5 + (2k)^3\tau^{-2} + (2k)\tau^{-4} \right) \vartheta.$$

Therefore

$$(C.2) \quad |L_{\partial\Omega}(\mathcal{A}) - L_{\partial\Omega}(\mathcal{A}')| \leq C(2k) \left((2k)^5 + (2k)^3\tau^{-2} + (2k)\tau^{-4} \right) \vartheta.$$

Remember that by Lemma 5.3 there exists a 1-Lipschitz function $\zeta^* : \cup_{i=1, \dots, k} \{p'_i, n'_i\} \rightarrow \mathbb{R}$ such that

$$L_{\partial\Omega}(\mathcal{A}') = \sum_{i=1}^k \zeta^*(p'_i) - \zeta^*(n'_i).$$

Define the function ζ for $d_{\partial\Omega}$ as in Definition 5.2, i.e. set

$$\zeta(x) := \max_{i \in \{1, \dots, k\}} (\zeta^*(p'_i) - d_i(x, \partial\Omega)),$$

where

$$d_i(x, \partial\Omega) := \min \left[\max \left(\max_{j \in \{1, \dots, 2k\}} d_{(i,j)}(x), d(p'_i, \partial\Omega) - d(x, \partial\Omega) \right), d(p'_i, \partial\Omega) + d(x, \partial\Omega) \right]$$

and

$$d_{(i,j)}(x) := \langle p'_i - x, \nu_{(i,j)} \rangle, \quad \nu_{(i,j)} = \begin{cases} \frac{p'_i - a'_j}{|p'_i - a'_j|} & \text{if } p'_i \neq a'_j \\ 0 & \text{if } p'_i = a'_j \end{cases}.$$

Lemma 5.4 yields that $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a 1-Lipschitz function such that

$$\sum_{i=1}^k \zeta(p'_i) - \zeta(n'_i) = \sum_{i=1}^k \zeta^*(p'_i) - \zeta^*(n'_i) = L_{\partial\Omega}(\mathcal{A}').$$

Recall that by Lemma C.1, letting

$$\Omega_{\mathcal{X}'} := \cap_{1 \leq l \leq n} \{z \mid \langle z - y_l, \nu(y_l) \rangle < 0\},$$

where $\nu(y_l)$ is the outer unit normal to $\partial\Omega$ at y_l , we have

$$(C.3) \quad |d(x, \partial\Omega_{\mathcal{X}'}) - d(x, \partial\Omega)| \leq C\tau^2 \quad \text{for any } x \in \mathbb{R}^3.$$

Observe that, since Ω is convex, $\Omega \subset \Omega_{\mathcal{X}'}$ and that for any $x \in \overline{\Omega}_{\mathcal{X}'}$

$$d(x, \partial\Omega_{\mathcal{X}'}) = \min_{1 \leq l \leq n} \langle y_l - x, \nu(y_l) \rangle.$$

In order to take advantage of this fact, we define a new function by replacing the distance to $\partial\Omega$ with the distance to $\partial\Omega_{\mathcal{X}'}$. More precisely, we let

$$\tilde{\zeta}(x) := \max_{i \in \{1, \dots, k\}} (\zeta^*(p'_i) - d_i(x, \partial\Omega_{\mathcal{X}'})).$$

From (C.3), we deduce that

$$(C.4) \quad \left| L_{\partial\Omega}(\mathcal{A}') - \sum_{i=1}^k \tilde{\zeta}(p'_i) - \tilde{\zeta}(n'_i) \right| \leq C(2k)\tau^2.$$

Next, we regularize the function $\tilde{\zeta}$. Let $\varphi \in C_c^\infty(B(0, 1), \mathbb{R}_+)$ be a mollifier such that $\int_{\mathbb{R}^3} \varphi(x) dx = 1$. Letting

$$(C.5) \quad \lambda := \vartheta^{1/\rho} \quad \text{for } \rho \in (0, 1/2),$$

we define

$$\zeta_\lambda(\cdot) := \varphi_\lambda * \tilde{\zeta}(\cdot) = \int_{\mathbb{R}^3} \varphi_\lambda(\cdot - z) \tilde{\zeta}(z) dz \quad \text{with } \varphi_\lambda(\cdot) = \frac{1}{\lambda} \varphi\left(\frac{\cdot}{\lambda}\right).$$

Argument for the first statement. Observe that $\|\tilde{\zeta} - \zeta_\lambda\|_{L^\infty(\mathbb{R}^3)} \leq \lambda$. We deduce that

$$(C.6) \quad \left| \sum_{i=1}^k \tilde{\zeta}(p'_i) - \tilde{\zeta}(n'_i) - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq 2k\lambda.$$

By combining (C.2) with (C.4) and (C.6), we obtain

$$(C.7) \quad \left| L_{\partial\Omega}(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq C(((2k)^6 + (2k)^4\tau^{-2} + (2k)^2\tau^{-4})\vartheta + 2k(\tau^2 + \lambda)).$$

On the other hand, note that

$$\begin{aligned} \left| \sum_{i=1}^k \zeta(p_i) - \zeta(n_i) - \sum_{i=1}^k \zeta(p'_i) - \zeta(n'_i) \right| &\leq \sum_{i=1}^k |p_i - n_i| + |p'_i - n'_i| \\ &\leq C(2k) ((2k)^5 + (2k)^3\tau^{-2} + (2k)\tau^{-4}) \vartheta. \end{aligned}$$

By combining the previous estimate with (C.4) and (C.6), we get

$$(C.8) \quad \left| \sum_{i=1}^k \zeta(p_i) - \zeta(n_i) - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq C(((2k)^6 + (2k)^4\tau^{-2} + (2k)^2\tau^{-4})\vartheta + 2k(\tau^2 + \lambda)).$$

Then from (C.7) and (C.8), we deduce that

$$\left| L_{\partial\Omega}(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p_i) - \zeta_\lambda(n_i) \right| \leq C(((2k)^6 + (2k)^4\tau^{-2} + (2k)^2\tau^{-4})\lambda^\rho + 2k\tau^2).$$

Argument for the second statement. Note that

$$(C.9) \quad \nabla\zeta_\lambda(x) = \int_{B(x,\lambda)} \varphi_\lambda(x-z) \nabla\tilde{\zeta}(z) dz$$

for any $x \in \mathbb{R}^3$. We define

$$\Lambda := \{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq 2k, i \neq j\} \quad \text{and} \quad c := \max_{i \in \{1, \dots, k\}} (\zeta^*(p'_i) - d(p'_i, \partial\Omega_{\mathcal{X}'})).$$

Then, letting

$$\begin{aligned} \zeta_{(i,j)}(\cdot) &:= \zeta^*(p'_i) - d_{(i,j)}(\cdot) \quad \text{for } (i, j) \in \Lambda, \\ \zeta_{l,+}(\cdot) &:= c + \langle \cdot - y_l, \nu(y_l) \rangle \quad \text{for } l \in \{1, \dots, n\}, \\ \zeta_{l,-}(\cdot) &:= c - \langle \cdot - y_l, \nu(y_l) \rangle \quad \text{for } l \in \{1, \dots, n\}, \end{aligned}$$

observe that, for almost every $z \in \Omega_{\mathcal{X}'}$,

$$\nabla\tilde{\zeta}(z) = \begin{cases} \nu_{(i,j)} & \text{if } \tilde{\zeta}(z) = \zeta_{(i,j)}(z) \text{ for some } (i, j) \in \Lambda \\ \nu(y_l) & \text{if } \tilde{\zeta}(z) = \zeta_{l,+}(z) \text{ for some } l \in \{1, \dots, n\} \\ -\nu(y_l) & \text{if } \tilde{\zeta}(z) = \zeta_{l,-}(z) \text{ for some } l \in \{1, \dots, n\}. \end{cases}$$

In particular $|\nabla\tilde{\zeta}(z)| = 1$ for almost every $z \in \Omega_{\mathcal{X}'}$. Thus

$$|\nabla\zeta_\lambda(x)| \leq \int_{B(x,\lambda)} \varphi_\lambda(x-z) |\nabla\tilde{\zeta}(z)| dz \leq \int_{B(x,\lambda)} \varphi_\lambda(x-z) dz = 1$$

for any $x \in \Omega_\lambda = \{x \in \Omega \mid 2\lambda < d(x, \partial\Omega)\}$.

Argument for the third statement. Observe that

$$\tilde{\zeta}(x) = c \quad \text{for any } x \in \partial\Omega_{\mathcal{X}'}$$

Thus

$$|\zeta_\lambda(\Omega \setminus \Omega_\lambda)| \leq C(\tau^2 + \lambda).$$

Argument for the fourth statement. Let us first prove that there exists a set $P_\lambda \subset \mathbb{R}^3$ such that $|\zeta_\lambda(P_\lambda)| \leq 2\lambda k^2$ and that, for any $0 < \kappa < \vartheta^2/3$,

$$C_\kappa := \{x \in \Omega_\lambda \mid |\nabla\zeta_\lambda(x)| < \kappa\} \setminus P_\lambda$$

can be covered by \mathfrak{B}_κ , a collection of at most $C((2k)^8 + \tau^{-8})$ balls of radius $C\lambda/(\vartheta^2 - 3\kappa)$. We will then conclude the desired result from this.

We start by defining the set P_λ . We let

$$P_\lambda := \{z \in \mathbb{R}^3 \mid d(z, P) \leq 2\lambda\},$$

where

$$P := \cup_{1 \leq i < j \leq k} P_{i,j}, \quad P_{i,j} := \{z \in \mathbb{R}^3 \mid \zeta_{(i,j)}(z) = \zeta_{(j,i)}(z)\}.$$

Arguing as in the proof of Proposition 5.1, we deduce that $|\zeta_\lambda(P_\lambda)| \leq C\lambda k^2$.

Let us now give a characterization of the gradient of ζ_λ for points $x \in \Omega_\lambda$. From (C.9), we deduce that

$$\nabla \zeta_\lambda(x) = \sum_{\alpha \in \Lambda} \sigma_\alpha \nu_\alpha + \sum_{l=1}^n \sigma_{l,+} \nu(y_l) + \sum_{l=1}^n \sigma_{l,-} (-\nu(y_l)),$$

where

$$\begin{aligned} \sigma_\alpha &= \int_{B(x,\lambda)} \varphi_\lambda(x-z) \mathbf{1}_{\tilde{\zeta}(z)=\zeta_\alpha(z)} dz \quad \text{for } \alpha \in \Lambda, \\ \sigma_{l,+} &= \int_{B(x,\lambda)} \varphi_\lambda(x-z) \mathbf{1}_{\tilde{\zeta}(z)=\zeta_{l,+}(z)} dz \quad \text{for } l \in \{1, \dots, n\}, \\ \sigma_{l,-} &= \int_{B(x,\lambda)} \varphi_\lambda(x-z) \mathbf{1}_{\tilde{\zeta}(z)=\zeta_{l,-}(z)} dz \quad \text{for } l \in \{1, \dots, n\}. \end{aligned}$$

Observe that $\sigma_\alpha, \sigma_{l,+}, \sigma_{l,-} \in [0, 1]$ and that $\sum_{\alpha \in \Lambda} \sigma_\alpha + \sum_{l=1}^n \sigma_{l,+} + \sum_{l=1}^n \sigma_{l,-} = 1$. We conclude that, for any $x \in \Omega_\lambda$, $\nabla \zeta_\lambda(x)$ can be written as a convex combination of the vectors ν_α 's, $\nu(y_l)$'s, and $-\nu(y_l)$'s with $\alpha \in \Lambda$, $l \in \{1, \dots, n\}$. By Caratheodory's theorem, we deduce that $\nabla \zeta_\lambda(x)$ can be written as a convex combination of at most four of them.

Now consider a number $0 < \kappa < \vartheta^2/3$ and a point $x \in C_\kappa$. We observe that, since $x \notin P_\lambda$, if there exists a point $y \in \overline{B(x, \lambda)}$ and indices $i, j \in \{1, \dots, k\}$ with $i \neq j$ such that

$$\zeta(y) = \zeta_{(i,j)}(y)$$

then, for any $z \in \overline{B(x, \lambda)}$,

$$\zeta(z) \neq \zeta_{(j,i)}(z).$$

On the other hand, since $x \in \Omega_\lambda$, if there exist a point $z_+ \in \overline{B(x, \lambda)}$ and an index $l \in \{1, \dots, n\}$ such that

$$\tilde{\zeta}(z_+) = \zeta_{l,+}(z_+)$$

then, for any $z \in \overline{B(x, \lambda)}$,

$$\tilde{\zeta}(z) \neq \zeta_{l,-}(z).$$

Arguing by contradiction, assume that there exist points $z_+, z_- \in \overline{B(x, \lambda)}$ and an index $l \in \{1, \dots, n\}$ such that

$$\tilde{\zeta}(z_+) = \zeta_{l,+}(z_+) \quad \text{and} \quad \tilde{\zeta}(z_-) = \zeta_{l,-}(z_-).$$

Observe that

$$|\tilde{\zeta}(z_+) - \tilde{\zeta}(z_-)| \leq \lambda \quad \text{and} \quad |\zeta_{l,-}(z_-) - \zeta_{l,-}(z_+)| \leq \lambda$$

and that

$$|\zeta_{l,+}(z_+) - \zeta_{l,-}(z_+)| = 2d(z_+, \partial\Omega_{\mathcal{X}'}) \geq 2d(z_+, \partial\Omega) > 2\lambda,$$

but

$$|\zeta_{l,+}(z_+) - \zeta_{l,-}(z_+)| = |\tilde{\zeta}(z_+) - \tilde{\zeta}(z_-) + \zeta_{l,-}(z_-) - \zeta_{l,-}(z_+)| \leq 2\lambda,$$

which yields a contradiction with the previous computation.

Analogously, if there exist a point $z_- \in \overline{B(x, \lambda)}$ and an index $l \in \{1, \dots, n\}$ such that

$$\tilde{\zeta}(z_-) = \zeta_{l,-}(z_-)$$

then, for any $z \in \overline{B(x, \lambda)}$,

$$\tilde{\zeta}(z) \neq \zeta_{l,+}(z).$$

This implies that $\nabla\zeta_\lambda(x)$ can be written as a convex combination of at most four vectors, where if one of them happens to be $\nu_{(i,j)}$ for some $i, j \in \{1, \dots, k\}$ with $i \neq j$ then all the other vectors are different from $\nu_{(j,i)} = -\nu_{(i,j)}$ and if one of them happens to be $\nu(y_l)$ (respectively $-\nu(y_l)$) for some $l \in \{1, \dots, n\}$ then all the other vectors are different from $-\nu(y_l)$ (respectively $\nu(y_l)$). Recalling that by Proposition C.1, we have

$$|v_1 \times v_2| \geq \vartheta \quad \text{and} \quad |\det(v_1, v_2, v_3)| \geq \vartheta^2$$

for any $v_1, v_2, v_3 \in \{\nu_{(i,j)} \mid 1 \leq i \leq k, 1 \leq j \leq 2k, i < j\} \cup \{\nu(y_l) \mid 1 \leq l \leq n\}$ with $v_1 \neq v_2 \neq v_3$, we deduce that $\nabla\zeta_\lambda(x)$ can be written as a convex combination of at most four vector that satisfy the previous property.

Arguing as in the proof of Proposition 5.1, we conclude that, since $x \in C_\kappa$ and $\kappa < \vartheta^2/3$, there exist four different functions

$$\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \{\zeta_{(i,j)} \mid (i,j) \in \Lambda\} \cup \{\zeta_{l,+} \mid y_l \in \mathcal{X}'\} \cup \{\zeta_{l,-} \mid y_l \in \mathcal{X}'\},$$

where if $\zeta_a = \zeta_{(i,j)}$ for some $(i,j) \in \Lambda$ and $a \in \{1, 2, 3, 4\}$ then $\zeta_b \neq \zeta_{(j,i)}$ for any $b \in \{1, 2, 3, 4\} \setminus \{a\}$ and if $\zeta_a = \zeta_{l,+}$ (respectively $\zeta_a = \zeta_{l,-}$) for some $l \in \{1, \dots, n\}$ and $a \in \{1, 2, 3, 4\}$ then $\zeta_b \neq \zeta_{l,-}$ (respectively $\zeta_b \neq \zeta_{l,+}$) for any $b \in \{1, 2, 3, 4\} \setminus \{a\}$, such that

$$\nabla\zeta_\lambda(x) = \sum_{i=1}^4 \alpha_i \zeta'_i$$

for some $\alpha_i \in (0, 1)$ with $\sum_{i=1}^4 \alpha_i = 1$, and

$$|\det(\zeta'_1 - \zeta'_2, \zeta'_3 - \zeta'_2, \zeta'_4 - \zeta'_2)| \geq 3(\vartheta^2 - 3\kappa).$$

Therefore the unique solution $z \in \mathbb{R}^3$ to the linear system of equations

$$\zeta_1(z) = \zeta_2(z) = \zeta_3(z) = \zeta_4(z)$$

lies in the ball $B(x, C\lambda/(\vartheta^2 - 3\kappa))$. We deduce that C_κ can be covered by \mathfrak{B}_κ , a collection of at most $\binom{|\Lambda|+2|\mathcal{X}'|}{4} \leq C((2k)^8 + \tau^{-8})$ balls of radius $C\lambda/(\vartheta^2 - 3\kappa)$.

Observing that

$$|D^2\zeta_\lambda(x)| \leq \frac{C}{\lambda^2}$$

for any $x \in \Omega_\lambda$, and letting

$$T_\kappa := \zeta_\lambda(\cup_{B \in \mathfrak{B}_\kappa} B) \cup \zeta_\lambda(P_\lambda),$$

we conclude that

$$|T_\kappa| \leq C((2k)^8 + \tau^{-8}) \frac{\lambda}{\vartheta^2 - 3\kappa}$$

and that, for any $t \in \zeta_\lambda(\Omega_\lambda) \setminus T_\kappa$, $\{x \mid \zeta_\lambda(x) = t\}$ is a complete submanifold of \mathbb{R}^3 whose second fundamental form is bounded by

$$C \frac{\sup_{\Omega_\lambda} |D^2\zeta_\lambda|}{\inf_{\Omega_\lambda \setminus ((\cup_{B \in \mathfrak{B}_\kappa} B) \cup P_\lambda)} |\nabla\zeta_\lambda|} \leq \frac{C}{\lambda^2\kappa}.$$

Recalling the relation between λ and ϑ (see (C.5)), the proposition follows. \square

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